Nonlinear steepest descent and the numerical solution of Riemann–Hilbert problems

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Abstract

The effective and efficient numerical solution of Riemann–Hilbert problems has been demonstrated in recent work. With the aid of ideas from the method of nonlinear steepest descent for Riemann–Hilbert problems, the resulting numerical methods have been shown numerically to retain accuracy as values of certain parameters become arbitrarily large. Remarkably, this numerical approach does not require knowledge of local parametrices; rather, the deformed contour is scaled near stationary points at a specific rate. The primary aim of this paper is to prove that this observed asymptotic accuracy is indeed achieved. To do so, we first construct a general theoretical framework for the numerical solution of Riemann–Hilbert problems. Second, we demonstrate the precise link between nonlinear steepest descent and the success of numerics in asymptotic regimes. In particular, we prove sufficient conditions for numerical methods to retain accuracy. Finally, we compute solutions to the homogeneous Painlevé II equation and the modified Korteweg–de Vries equations to explicitly demonstrate the practical validity of the theory.

1 Introduction

Matrix-valued Riemann–Hilbert problems (RHPs) are of profound importance in modern applied analysis. In inverse scattering theory, solutions to the nonlinear Schrödinger equation, Korteweg de–Vries equation (KdV), Kadomtsev–Petviashvili I equation and many other integrable solutions can be written in terms of solutions of RHPs [1]. Orthogonal polynomials can also be rewritten in terms of solutions of RHPs [3]. The asymptotics of orthogonal polynomials is crucial for determining the distribution of eigenvalues of large random matrix ensembles [3]. In each of these applications, RHPs fulfill the role that integral representations play in classical asymptotic analysis.

The way in which these RHPs are analyzed is through the method of nonlinear steepest descent [4]. RHPs can be deformed in the complex plane in much the same way as contour integrals. This allows the oscillatory nature of the problem to be changed to exponential decay. These deformed problems, which depend on a parameter, are solved approximately. The approximate solution is found explicitly and the

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difference between the actual solution and this approximate solution tends to zero as a parameter becomes large.

The method of nonlinear steepest descent was adapted by the authors in [15] for numerical purposes. We developed a method to reliably solve the Cauchy initial-value problem for KdV and modified KdV for all values of x and t. A benefit of the approach was that, unlike the standard method of nonlinear steepest descent, we did not require the knowledge of difficult-to-derive local parametrices. Instead, an approach was developed based on scaling contours. Convergence of the approximation was demonstrated through numerical tests, which further showed that the accuracy of the approximation actually increased in the asymptotic regime. The focus of the current paper is to derive sufficient conditions (which are often satisfied) for which we can prove that the approach of solving RHPs numerically on scaled contours will be guaranteed to be accurate in asymptotic regimes. We refer to this type of behavior as asymptotic stability or uniform approximation.

In addition, we show the deep connection between the success of the numerical method and the success of the method of nonlinear steepest descent [4]. A notable conclusion is that one can expect that whenever the method of nonlinear steepest descent produces an asymptotic formula, the numerical method can be made asymptotically stable. Achieving this requires varying amounts of preconditioning of the RHP. This can vary from not deforming the RHP at all, all the way to using the full deformation needed by the analytical method. An important question is: "when can we stop deformations and a have a reliable numerical method?" Our main results are in §5 and allow us to answer this question. In short, although we do not require the knowledge of local parametrices to construct the numerical method, their existence ensures that the numerical method remains accurate, and their explicit knowledge allows us to analyze the error of the approximation directly. In the last two sections we provide useful examples of where this arises in applications.

The paper is structured as follows. We begin with some background material, followed by the precise definition of a RHP along with properties of an associated singular integral operator. This allows us, amongst other things, to address the regularity of solutions. Then, we use an abstract framework for the numerical solution of RHP which will allow us to address asymptotic accuracy in a more concise way. Additionally, other numerical methods (besides the one used for the applications) may fit within the framework of this paper. We review the philosophy and analysis behind nonlinear steepest descent and how it relates to our numerical framework. Then, we prove our main results which provide sufficient condition for uniform approximation. The numerical approach of [11] is placed within the general framework, along with necessary assumptions which allows a realization of uniform approximation. We apply the theory to two RHPs. The first is a RHP representation of the homogeneous Painlevé II transcendent

$$u_{xx} - xu - 2u^3 = 0, \tag{1.1}$$

for specific initial conditions and $x < 0$. The second is a RHP representation of the modified Korteweg–de Vries equation

$$u_t - 6u^2 u_x + u_{xxx} = 0, \tag{1.2}$$

for smooth exponentially decaying initial data in the so-called Painlevé region [5].

2 Background Material

We use this section to fix notation that will be used throughout the remainder of the manuscript. We reserve C and C_i to denote generic constants, and X and Y for Banach spaces. We denote the norm on a space X by $\|\cdot\|_X$. The notation $\mathcal{L}(X,Y)$ is used to denote the Banach algebra of all bounded linear operators from X to Y. When $X = Y$ we write $\mathcal{L}(X)$ to simplify notation. The following lemma is of great use [2]:

Lemma 2.1. *Assume $L \in \mathcal{L}(X,Y)$ has a bounded inverse $L^{-1} \in \mathcal{L}(Y,X)$. Assume $M \in \mathcal{L}(X,Y)$ satisfies*

$$\|M - L\| < \frac{1}{\|L^{-1}\|}.$$

Then M is invertible and

$$\|M^{-1}\| \leq \frac{\|L^{-1}\|}{1 - \|L^{-1}\|\|L - M\|}. \tag{2.1}$$

Furthermore,

$$\|L^{-1} - M^{-1}\| \leq \frac{\|L^{-1}\|^2\|L - M\|}{1 - \|L^{-1}\|\|L - M\|}. \tag{2.2}$$

In what follows we are interested in functions defined on oriented contours in $\mathbb{C}$. Assume Γ is piecewise smooth, oriented and (most likely) self-intersecting. γ_0 is used to denote the set of self-intersections. Decompose $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_\ell$ to its smooth, non-self-intersecting components. Define $C_c^\infty(\Gamma_i)$ to be the space of infinitely differentiable functions with compact support in Γ_i, and $C(\Gamma_i)$ to be the Banach space of continuous functions with the uniform norm. Define the space

$$L^2(\Gamma) = \left\{ f \text{ measurable} : \sum_{i=1}^{\ell} \int_{\Gamma_i} |f(k)|^2 |dk| < \infty \right\}.$$

We define, D, the distributional differentiation operator for functions defined on $\Gamma \setminus \gamma_0$. For a function $\varphi \in C_c^\infty(\Gamma_i)$ we represent a linear function g via the dual pairing

$$g(\varphi) = \langle g, \varphi \rangle_{\Gamma_i}.$$

To $D_i g$ we associate the functional

$$\langle g, \varphi_i' \rangle_{\Gamma_i}.$$

For $f \in L^2(\Gamma)$ consider $f_i = f|_{\Gamma_i}$, the restriction of f to Γ_i. In the case that the distribution $D_i f_i$ corresponds to a locally integrable function

$$\langle D_i f_i, \varphi \rangle_{\Gamma_i} = \int_{\Gamma_i} D_i f_i(k) \varphi(k) dk = \int_{\Gamma_i} f_i(k) \varphi'(k) dk,$$

for each i, we define

$$Df(k) = D_i f_i(k) \text{ if } k \in \Gamma_i \setminus \gamma_0.$$

This allows us to define

$$H^k(\Gamma) = \left\{ f \in L^2(\Gamma) : D^j f \in L^2(\Gamma), \quad j = 0, \ldots, k \right\}.$$

We write $W^{k,\infty}(\Gamma)$ for the Sobolev space with the L^2 norm replaced with the L^∞ norm. An important note is that we will be dealing with matrix-valued functions, and hence the definitions of all these spaces must be suitably extended. Since all finite-dimensional norms are equivalent, we can use the above definitions in conjunction with any matrix norm to define a norm for matrix-valued functions provided the norm is sub-additive.

3 Theory of Riemann–Hilbert Problems in $L^2(\Gamma)$

Loosely speaking, a Riemann–Hilbert problem (RHP) is a boundary-value problem in the complex plane:

Problem 3.1. *[10] Given an oriented contour $\Gamma \subset \mathbb{C}$ and a jump matrix $G : \Gamma \to \mathbb{C}^{2\times 2}$, find a bounded function $\Phi : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{2\times 2}$ which is analytic everywhere in the complex plane except on Γ, such that*

$$\Phi^+(z) = \Phi^-(z) G(z), \qquad \textit{for } z \in \Gamma, \textit{ and} \tag{3.1}$$

$$\Phi(\infty) = I, \tag{3.2}$$

where Φ^+ denotes the limit of Φ as z approaches Γ from the left, Φ^- denotes the limit of Φ as z approaches Γ from the right, and $\Phi(\infty) = \lim_{|z|\to\infty} \Phi(z)$. We denote this RHP by $[G; \Gamma]$.

The definition is not sufficiently precise to compute solutions. We say that Φ is an L^2 solution of $[G;\Gamma]$ normalized at infinity, if

$$\Phi(z) = I + \frac{1}{2\pi i}\int_\Gamma \frac{u(s)}{s-z} ds = I + \mathcal{C}_\Gamma u(z),$$

for some $u \in L^2(\Gamma)$ and $\Phi^+(z) = \Phi^-(z)G(z)$, for almost every $z \in \Gamma$.

Definition 3.1. *The* Cauchy transform pair *is the pair of boundary values of the Cauchy integral $\mathcal{C}_\Gamma$:*

$$\mathcal{C}_\Gamma^\pm u(z) = \left(\mathcal{C}_\Gamma u(z)\right)^\pm. \tag{3.3}$$

It is well known that for fairly general contours (including every contour we considered here) that the limits in (3.3) exist non-tangentially almost everywhere. The operators $\mathcal{C}_\Gamma^\pm$ are bounded from $L^2(\Gamma)$ to itself and satisfy the operator identity [8]

$$\mathcal{C}_\Gamma^+ - \mathcal{C}_\Gamma^- = I. \tag{3.4}$$

Remark 3.1. *When deriving a RHP one must show that $\Phi - I \in \operatorname{ran}\mathcal{C}_\Gamma$. This is generally done using Hardy space methods [8, 6]. Brevity is the governing motivation for the above definition of a RHP.*

We convert the RHP into an equivalent singular integral equation (SIE). Assume $\Phi(z) = I + \mathcal{C}_\Gamma u(z)$ and substitute into (3.1) to obtain

$$I + \mathcal{C}_\Gamma^+ u(z) = \mathcal{C}_\Gamma^- u(z)G(z) + G(z). \tag{3.5}$$

Using (3.4),

$$u(z) - \mathcal{C}_\Gamma^- u(z)(G(z) - I) = G(z) - I. \tag{3.6}$$

Definition 3.2. *We use $\mathcal{C}[G;\Gamma]$ to refer to the operator (bounded on $L^2(\Gamma)$ provided $G \in L^\infty(\Gamma)$) in (3.6).*

In what follows we assume at a minimum that the RHP is well-posed, or $\mathcal{C}[G;\Gamma]^{-1}$ exists and is bounded in $L^2(\Gamma)$ and $G - I \in L^2(\Gamma)$.

We need to establish the smoothness of solutions of (3.6) since we approximate solutions numerically. This smoothness relies on the smoothness of the jump matrix in the classical sense along with a type of smoothness at the self-intersection points, γ_0, of the contour.

Definition 3.3. *Assume $a \in \gamma_0$. Let $\Gamma_1, \ldots, \Gamma_m$ be a counter-clockwise ordering of subcomponents of Γ which contain $z = a$ as an endpoint. For $G \in W^{k,\infty}(\Gamma) \cap H^k(\Gamma)$ we define $\hat{G}_i$ by $G|_{\tilde{\Gamma}_i}$ if Γ_i is oriented outwards and $(G|_{\Gamma_i})^{-1}$ otherwise. We say G satisfies the $(k-1)$th-order product condition if using the $(k-1)$th-order Taylor expansion of each $\hat{G}_i$ we have*

$$\prod_{i=1}^m \hat{G}_i = I + \mathcal{O}\left((z-a)^k\right), \text{ for } j = 1, \ldots, k-1, \quad \forall a \in \gamma_0. \tag{3.7}$$

To capture all the regularity properties of the solution of (3.6), and to aid the development of numerical methods, we introduce the following definition in the same vein as Definition 3.3.

Definition 3.4. *Assume that $a \in \gamma_0$ and let $\Gamma_1, \ldots, \Gamma_m$ be a counter-clockwise ordering of subcomponents of Γ which contain $z = a$ as an endpoint. For $f \in H^k(\Gamma)$, define*

$$f_i^{(j)} = \begin{cases} -\lim_{z\to a}\left(\frac{d}{dz}\right)^j f|_{\Gamma_i}(z) & \text{if } \Gamma_i \text{ is oriented outward,} \\ \lim_{z\to a}\left(\frac{d}{dz}\right)^j f|_{\Gamma_i}(z) & \text{if } \Gamma_i \text{ is oriented inward.} \end{cases} \tag{3.8}$$

We say that f satisfies the $(k-1)$th-order zero-sum condition if

$$\sum_{i=1}^m f_i^{(j)} = 0, \text{ for } j = 0, \ldots, k-1 \text{ and } \forall a \in \gamma_0. \tag{3.9}$$

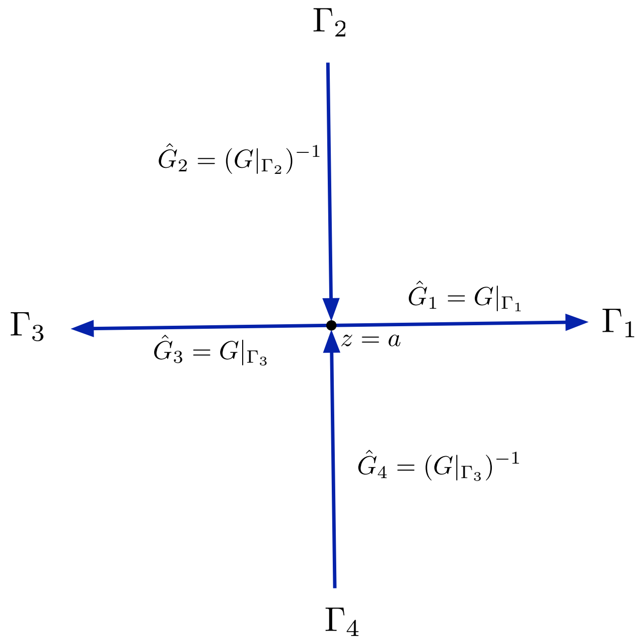

Figure 1: A typical intersection point and the definition of $\hat{G}_i$.

Remark 3.2. *These definitions imply conditions also when Γ has an isolated endpoint. In that case $G = I$ at this endpoint and $f = 0$.*

This motivates the following definition.

Definition 3.5. *A RHP $[G; \Gamma]$ is said to be k-regular if*

- *G, G^{-1} satisfy the $(k-1)$th-order product condition, and*
- *$G - I, G^{-1} - I \in W^{k,\infty}(\Gamma) \cap H^k(\Gamma)$.*

It is worth noting that when Γ is bounded the $H^k(\Gamma)$ condition is trivial. We include it for completeness. A useful result [9, Chapter II, Lemma 6.1] is:

Lemma 3.1. *Let Γ be a smooth non-closed curve from $z = a$ to $z = b$ with orientation from a to b. If $f \in H^1(\Gamma)$, then*

$$D \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{f(a)}{a - z} - \frac{f(b)}{b - z} + \int_\Gamma \frac{Df(\zeta)}{\zeta - z} d\zeta.$$

Due to cancellation, $\mathcal{C}_\Gamma^\pm$ commutes with the weak differentiation operator for functions that satisfy the zeroth-order zero-sum condition. We use the notation $H_z^k(\Gamma)$ for the closed subspace of $H^k(\Gamma)$ consisting of functions which satisfy the $(k-1)$th order zero-sum condition. This allows us to use the boundedness of $\mathcal{C}_\Gamma^\pm$ on $L^2(\Gamma)$ to show the boundedness (with same norm) of $\mathcal{C}_\Gamma^\pm$ from $H_z^k(\Gamma)$ to $H^k(\Gamma)$. The results in [16], combined with the fact that differentiation commutes with the Cauchy operator on these zero-sum spaces can be used to prove the following theorem.

Theorem 3.1. *Given a RHP $[G; \Gamma]$ which is k-regular, assume $\mathcal{C}[G; \Gamma]$ is invertible on $L^2(\Gamma)$. Then $u \in H_z^k(\Gamma)$, the solution of (3.6), satisfies*

$$D^k u = \mathcal{C}[G; \Gamma]^{-1} \left(D^k (G - I) + D^k (\mathcal{C}_\Gamma^- u \cdot (G - I)) - \mathcal{C}_\Gamma^- D^k u \cdot (G - I) \right). \tag{3.10}$$

where the right-hand side of (3.10) does not depend on $D^k u$.

Corollary 3.1. *Under the hypotheses of Theorem 3.1, u satisfies an inequality of the form*

$$\|u\|_{H^k(\Gamma)} \leq p_k \left(\|G - I\|_{W^{k,\infty}(\Gamma)} \|\mathcal{C}[G;\Gamma]^{-1}\|_{\mathcal{L}(L^2(\Gamma))} \right) \|\mathcal{C}[G;\Gamma]^{-1}\|_{\mathcal{L}(L^2(\Gamma))} \|G - I\|_{H^k(\Gamma)}, \tag{3.11}$$

where p_k is a polynomial of degree k whose coefficients depend on $\|\mathcal{C}_\Gamma^-\|_{\mathcal{L}(L^2(\Gamma))}$.

Proof: Taking the norm of (3.10) gives a bound on the semi-norm $\|D^k u\|_{L^2(\Gamma)}$ in terms of $\|u\|_{H^{k-1}(\Gamma)}$. Using (3.10) for $k-1$ gives a bound in terms of $\|u\|_{H^{k-2}(\Gamma)}$. This process produces a bound of the form (3.11). ■

Remark 3.3. *The expression for the derivative in Theorem 3.1 can be used to bound Sobolev norms of the solution in terms of Sobolev norms of the jump matrix if a bound on the norm of the inverse operator is known. In some cases, when the jump matrix depends on a parameter, and the Sobolev norms of the jump matrix are bounded or decaying, the resulting bounds are of use.*

We often use the following theorem which is derived from the results in [7].

Theorem 3.2. *Consider a sequence of RHPs $\{[G_\xi;\Gamma]\}_{\xi \geq 0}$ on the fixed contour Γ which are k-regular. Assume $G_\xi \to G$ in $W^{k,\infty}(\Gamma) \cap H^k(\Gamma)$ as $\xi \to \infty$, then*

- *If $\mathcal{C}[G;\Gamma]$ is invertible then there exists a $T > 0$ such that $\mathcal{C}[G_\xi;\Gamma]$ is also invertible for $\xi > T$.*
- *If Φ_ξ is the solution of $[G_\xi,\Gamma]$, and Φ the solution of $[G;\Gamma]$, then $\Phi_\xi^\pm - \Phi^\pm \to 0$ in $H^k(\Gamma)$.*
- *$\|\Phi_\xi - \Phi\|_{W^{j,\infty}(S)} \to 0$, for all $j \geq 1$, whenever S is bounded away from Γ.*

Proof: The first statement follows from the fact that $\mathcal{C}[G_\xi;\Gamma]$ converges to $\mathcal{C}[G;\Gamma]$ in operator norm. The second property follows from Corollary 3.1. The final property is a consequence of the Cauchy-Schwartz inequality and the fact that $\|u_\xi - u\|_{L^2(\Gamma)} \to 0$. ■

4 The Numerical Solution of Riemann–Hilbert Problems

The goal in this section is to introduce the necessary tools to approximate the operator equation

$$\mathcal{C}[G;\Gamma]u = G - I. \tag{4.1}$$

We start with two projections $\mathcal{I}_n$ and $\mathcal{P}_n$, both of finite rank. Assume $\mathcal{P}_n$ is defined on $H_z^1(\Gamma)$ and $\mathcal{I}_n$ is defined on $H^1(\Gamma)$. Define $X_n = \operatorname{ran} \mathcal{P}_n$ and $Y_n = \operatorname{ran} \mathcal{I}_n$ equipping both spaces with the inherited $L^2(\Gamma)$ norm. We obtain a finite-dimensional approximation of $\mathcal{C}[G;\Gamma]$ by defining

$$\mathcal{C}_n[G;\Gamma]u = \mathcal{I}_n \mathcal{C}[G;\Gamma]u.$$

It follows that $\mathcal{C}_n[G;\Gamma] : X_n \to Y_n$. This is useful under the assumption that we can compute $\mathcal{C}[G;\Gamma]$ exactly for all $u \in X_n$. An approximate solution of (4.1) is obtained by

$$u_n = \mathcal{C}_n[G;\Gamma]^{-1} \mathcal{I}_n (G - I),$$

whenever the operator is invertible. We use the pair $(\mathcal{I}_n, \mathcal{P}_n)$ to refer to this numerical method.

Remark 4.1. *One may have concerns about $\mathcal{C}_n[G;\Gamma]$ because the dimension of the domain and that of the range are different. It turns out that $\mathcal{C}[G;\Gamma]$ maps $H_z^1(\Gamma)$ to a closed subspace of $H^1(\Gamma)$ and we can define $\mathcal{I}_n$ on this space. In the numerical framework of [11], solving the associated linear system results in a solution that must satisfy the zeroth-order zero-sum condition, justifying the theoretical construction above. In what follows we ignore this detail.*

To simplify notation, we define $\mathcal{T}[G;\Gamma]u = \mathcal{C}_\Gamma^- u(G - I)$ (so that $\mathcal{C}[G;\Gamma] = I - \mathcal{T}[G;\Gamma]$) and $\mathcal{T}_n[G;\Gamma] = \mathcal{I}_n \mathcal{T}[G;\Gamma]$. We use a few definitions to describe the required properties of the projections.

Definition 4.1. *The approximation $\mathcal{C}_n[G;\Gamma]$ to $\mathcal{C}[G;\Gamma]$ is said to be of type (α,β,γ) if, whenever $\mathcal{C}[G;\Gamma]$ is invertible for $n>N$, $\mathcal{C}_n[G;\Gamma]$ is invertible and*

- $\|\mathcal{C}_n[G;\Gamma]\|_{\mathcal{L}(H_z^1(\Gamma),Y_n)} \leq C_1 n^\alpha(1+\|G-I\|_{L^\infty(\Gamma)}\|\mathcal{C}_\Gamma^-\|_{\mathcal{L}(L^2(\Gamma))})$,
- $\|\mathcal{C}_n[G;\Gamma]^{-1}\|_{\mathcal{L}(Y_n,X_n)} \leq C_2 n^\beta \|\mathcal{C}[G;\Gamma]^{-1}\|_{\mathcal{L}(L^2(\Gamma))}$ *and*
- $\|\mathcal{T}_n[G;\Gamma]\|_{\mathcal{L}(X_n,Y_n)} \leq C_3 n^\gamma \|G-I\|_{L^\infty(\Gamma)}\|\mathcal{C}_\Gamma^-\|_{\mathcal{L}(L^2(\Gamma))}$.

The constants here are allowed to depend on Γ.

The first and second conditions in Definition 4.1 are necessary for the convergence of the numerical method. This will be made more precise below. The first and third conditions are needed to control operator norms as G changes. It is not surprising that the first and the third conditions are intimately related and in §6 we demonstrate the connection.

Remark 4.2. *Some projections, mainly those used in Galerkin methods, can be defined directly on $L^2(\Gamma)$. In this case we replace the first condition in Definition 4.1 with*

$$\|\mathcal{C}_n[G;\Gamma]\|_{\mathcal{L}(L^2(\Gamma),Y_n)} \leq C_1 n^\alpha(1+\|G-I\|_{L^\infty(\Gamma)}\|\mathcal{C}_\Gamma^-\|_{\mathcal{L}(L^2(\Gamma))}).$$

This condition and the second condition with $\alpha=\gamma$ are implied by requiring

$$\|\mathcal{I}_n\|_{\mathcal{L}(L^2(\Gamma),Y_n)} \leq C_1 n^\alpha.$$

In this sense Galerkin methods are more natural for RHPs, though we use the collocation method of [11] below because a Galerkin method has yet to be developed.

Definition 4.2. *The pair $(\mathcal{I}_n,\mathcal{P}_n)$ is said to produce an admissible numerical method if*

- *The method is of type (α,β,γ).*
- *For all $m>0$, $\|\mathcal{I}_n u-u\|_{H^1(\Gamma)}$ and $\|\mathcal{P}_n u-u\|_{H^1(\Gamma)}$ tend to zero faster than n^{-m} as $n\to\infty$ for all $u\in H^s(\Gamma)$; for some s.*
- *$\mathcal{I}_n$ is bounded from $C(\Gamma)$ to $L^2(\Gamma)$, uniformly in n.*

Remark 4.3. *We assume spectral convergence of the projections. This can be relaxed but one has to spend considerable effort to ensure α, β and γ are sufficiently small.*

Next, we prove the generalized convergence theorem.

Theorem 4.1. *Assume that $(\mathcal{I}_n,\mathcal{P}_n)$ produces an admissible numerical method. If $\mathcal{C}[G;\Gamma]$ is 1-regular and invertible on $L^2(\Gamma)$, we have*

$$\|u-u_n\|_{L^2(\Gamma)} \leq (1+cn^{\alpha+\beta})\|\mathcal{P}_n u-u\|_{H^1(\Gamma)} \quad \textit{with} \tag{4.2}$$
$$c = C\|\mathcal{C}[G;\Gamma]^{-1}\|_{\mathcal{L}(L^2(\Gamma))}(1+\|G-I\|_{L^\infty(\Gamma)}\|\mathcal{C}_\Gamma^-\|_{\mathcal{L}(L^2(\Gamma))}). \tag{4.3}$$

Proof: First, for notational simplicity, define $\mathcal{K}_n=\mathcal{C}_n[G;\Gamma]$, $\mathcal{K}=\mathcal{C}[G;\Gamma]$ and $f=G-I$. Then $u_n = \mathcal{K}_n^{-1}\mathcal{I}_n f = \mathcal{K}_n^{-1}\mathcal{I}_n\mathcal{K}u$. Further, since $u\in H_z^1(\Gamma)$,

$$\begin{aligned}
u-u_n =& u-\mathcal{P}_n u+\mathcal{P}_n u-u_n\\
=& u-\mathcal{P}_n u+\mathcal{P}_n u-\mathcal{K}_n^{-1}\mathcal{I}_n\mathcal{K}u\\
=& u-\mathcal{P}_n u+\mathcal{K}_n^{-1}\mathcal{K}_n\mathcal{P}_n u-\mathcal{K}_n^{-1}\mathcal{I}_n\mathcal{K}u\\
=& u-\mathcal{P}_n u+\mathcal{K}_n^{-1}(\mathcal{K}_n\mathcal{P}_n u-\mathcal{I}_n\mathcal{K}u)\\
=& u-\mathcal{P}_n u+\mathcal{K}_n^{-1}\mathcal{I}_n\mathcal{K}(\mathcal{P}_n u-u).
\end{aligned}$$

We used $\mathcal{K}_n\mathcal{P}_n u=\mathcal{I}_n\mathcal{K}\mathcal{P}_n u$ for $u\in H_z^1(\Gamma)$ in the last line. Taking an $L^2(\Gamma)$ norm, we have

$$\|u-u_n\|_{L^2(\Gamma)} \leq \|(I+\mathcal{K}_n^{-1}\mathcal{I}_n\mathcal{K}(u-\mathcal{P}_n u)\|_{L^2(\Gamma)} \leq (1+cn^{\alpha+\beta})\|u-\mathcal{P}_n u\|_{H^1(\Gamma)}.\blacksquare$$

Remark 4.4. *In the case mentioned above where $\mathcal{I}_n$ and $\mathcal{P}_n$ can be defined directly on $L^2(\Gamma)$ we obtain a purely $L^2(\Gamma)$ based bound*

$$\|u - u_n\|_{L^2(\Gamma)} \leq (1 + cn^{\alpha+\beta})\|u - \mathcal{P}_n u\|_{L^2(\Gamma)}.$$

Corollary 4.1. *Under the assumptions of Theorem 4.1 and assuming that $[G;\Gamma]$ is k-regular for large k (large is determined by Definition 4.2) we have that $\Phi_n = I + \mathcal{C}_\Gamma u_n$ is an approximation of Φ, the solution of $[G;\Gamma]$, in the following sense.*

- $\Phi_n^\pm - \Phi^\pm \to 0$ *in* $L^2(\Gamma)$ *and*
- $\|\Phi_n - \Phi\|_{W^{j,\infty}(S)} \to 0$ *for all* $j \geq 0$, *whenever S is bounded away from Γ.*

Proof: The first claim follows from the boundedness of the Cauchy operator on $L^2(\Gamma)$ and, as before, the Cauchy-Schwartz inequality gives the second. ■

Below we always assume the numerical method considered is admissible. The ideas presented thus far are general. In specific cases the contour Γ consists of disjoint components. We take a different approach to solving the RHP in this case.

Example 4.1. *Consider the RHP $[G;\Gamma]$ with $\Gamma = \Gamma_1 \cup \Gamma_2$ where Γ_1 and Γ_2 are disjoint. To solve the full RHP, we first solve for Φ_1 — the solution of $[G|_{\Gamma_1};\Gamma_1]$ — assuming that this sub-problem has a unique solution. The jump on Γ_2 is modified through conjugation by Φ_1. Define*

$$\tilde{G}_2 = \Phi_1 G|_{\Gamma_2} \Phi_1^{-1}.$$

The solution Φ_2 of $[\tilde{G}_2;\Gamma_2]$ is then found. A simple calculation shows that $\Phi = \Phi_1\Phi_2$ solves the original RHP $[G;\Gamma]$.

This idea allows us to treat each disjoint contour separately, solving in an iterative way. When using this algorithm numerically, the dimension of the linear system solved at each step is a fraction of that of the full discretized problem. This produces significant computational savings. We now generalize these ideas.

Consider a RHP $[G;\Gamma]$ where $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_\ell$. Here each Γ_i is disjoint and $\Gamma_i = \alpha_i\Omega_i + \beta_i$ for some contour Ω_i. We define $G_i(z) = G(z)|_{\Gamma_i}$ and $H_i(k) = G_i(\alpha_i k + \beta_i)$. As a notational remark, we always associate H_i and G is this way.

Remark 4.5. *The motivation for introducing the representation of the contours in this fashion will be made clear below. Mainly, this formulation is important when α_i and/or β_i depend on a parameter but Ω_i does not.*

We now describe the general iterative solver.

Algorithm 4.1. *(Scaled and Shifted RH Solver)*

1. *Solve the RHP $[H_1;\Omega_1]$ to obtain $\tilde{\Phi}_1$. We denote the solution of the associated SIE as U_1 with domain Ω_1. Define $\Phi_1(z) = \tilde{\Phi}_1\left(\frac{z-\beta_1}{\alpha_1}\right)$.*
2. *For each $j = 2, \ldots, \ell$ define $\Phi_{i,j}(z) = \Phi_i(\alpha_j z + \beta_j)$ and solve the RHP $[\tilde{H}_j;\Omega_j]$ with*
$$\tilde{H}_j = \Phi_{j-1,j} \cdots \Phi_{1,j} H_j \Phi_{1,j}^{-1} \cdots \Phi_{j-1,j}^{-1},$$
to obtain $\tilde{\Phi}_j$. Again, the solution of the integral equation is denoted by U_j with domain Ω_j. Define $\Phi_j(z) = \tilde{\Phi}_j\left(\frac{z-\beta_j}{\alpha_j}\right)$.
3. *Construct $\Phi = \Phi_\ell \cdots \Phi_1$, which satisfies the original problem.*

When this algorithm is implemented numerically, the jump matrix corresponding to $\tilde{H}_j$ is not exact. It depends on the approximations of each of the Φ_i for $i < j$ and more specifically, it depends on the order of approximation of the RHP on Ω_i for $i < j$. We use the notation $\boldsymbol{n}_i = (n_1, \ldots, n_i)$ where each n_i is the order of approximation on Ω_i. We use $\boldsymbol{n} > \boldsymbol{m}$ whenever the vectors are of the same length and $n_j > m_j$ for all j. The statement $\boldsymbol{n} \to \infty$ means that each component of $\boldsymbol{n}$ tends to ∞. Let $\Phi_{i,j,\boldsymbol{n}_i}$ be the approximation of $\Phi_{i,j}$ and define

$$\tilde{H}_{j,\boldsymbol{n}_j} = \Phi_{j-1,j,\boldsymbol{n}_{j-1}} \cdots \Phi_{1,j,\boldsymbol{n}_1} H_j \Phi_{1,j,\boldsymbol{n}_1}^{-1} \cdots \Phi_{j-1,j,\boldsymbol{n}_{j-1}}^{-1}.$$

If the method converges then $\tilde{H}_{j,\boldsymbol{n}_j} \to \tilde{H}_j$ uniformly as $\boldsymbol{n}_j \to \infty$.

A significant remaining question is: "how do we know solutions exist at each stage of this algorithm?" In general, this is not the case. $\mathcal{C}[G;\Gamma]$ can be expressed in the form $\mathcal{K} - \mathcal{T}$ where $\mathcal{K}$ is the block-diagonal operator with blocks $\mathcal{C}[G_i;\Gamma_i]$ and $\mathcal{T}$ is a compact operator. Here $\mathcal{T}$ represents the effect of one contour on another and if the operator norm of $\mathcal{T}$ is sufficiently small solutions exist at each iteration of Algorithm 4.1. This is true if the arclength of each Γ_i is sufficiently small. We leave a more thorough discussion of this to §5.1. An implicit assumption in our numerical framework is that such equations are uniquely solvable.

The final question is one of convergence. For a single fixed contour we know that if $(\mathcal{I}_n, \mathcal{P}_n)$ produces an admissible numerical method and the RHP is sufficiently regular, the numerical method converges. This means that the solution of this RHP converges uniformly, away from the contour it is defined on. This is the basis for proving that Algorithm 4.1 converges. Theorem 3.2 aids us when considering the infinite-dimensional operator for which the jump matrix is uniformly close, but we need an additional result for the finite-dimensional case.

Lemma 4.1. *Consider a sequence of RHPs $\{[G_\xi;\Gamma]\}_{\xi\geq 0}$ on the fixed contour Γ which are k-regular. Assume $G_\xi \to G$ in $L^\infty(\Gamma) \cap L^2(\Gamma)$ as $\xi \to \infty$ and $[G;\Gamma]$ is k-regular, then*

- *If $\mathcal{C}_n[G;\Gamma]$ is invertible, then there exists $T(n) > 0$ such that $\mathcal{C}_n[G_\xi;\Gamma]$ is also invertible for $\xi > T(n)$.*
- *If $\Phi_{n,\xi}$ is the approximate solution of $[G_\xi;\Gamma]$ and Φ_n is the approximate solution of $[G;\Gamma]$, then $\Phi_{n,\xi} - \Phi_n \to 0$ in $L^2(\Gamma)$ as $\xi \to \infty$ for fixed n.*
- *$\|\Phi_{n,\xi} - \Phi_n\|_{W^{j,\infty}(S)} \to 0$, as $\xi \to \infty$, for all $j \geq 1$, whenever S is bounded away from Γ for fixed n.*

Proof: We consider the two equations

$$\mathcal{C}_n[G_\xi;\Gamma]u_{n,\xi} = \mathcal{I}_n(G_\xi - I),$$
$$\mathcal{C}_n[G;\Gamma]u_n = \mathcal{I}_n(G - I).$$

Since the method is of type (α,β,γ), we have (see Definition (4.1)),

$$\|\mathcal{C}_n[G_\xi;\Gamma] - \mathcal{C}_n[G;\Gamma]\|_{\mathcal{L}(X_n,Y_n)} \leq C_3 n^\gamma \|\mathcal{C}_\Gamma^-\|_{\mathcal{L}(L^2(\Gamma))}\|G_\xi - G\|_{L^\infty(\Gamma)} = E(\xi)n^\gamma.$$

For fixed n, by increasing ξ, we can make $E(\xi)$ small, so that

$$\|\mathcal{C}_n[G_\xi;\Gamma] - \mathcal{C}_n[G;\Gamma]\|_{\mathcal{L}(X_n,Y_n)} \leq \frac{1}{C_2}\frac{1}{\|\mathcal{C}[G;\Gamma]^{-1}\|_{\mathcal{L}(L^2(\Gamma))}} n^{-\beta} \leq \frac{1}{\|\mathcal{C}_n[G;\Gamma]^{-1}\|_{\mathcal{L}(Y_n,X_n)}}.$$

Specifically, we choose ξ small enough so that

$$E(\xi) \leq \frac{1}{2}\frac{1}{C_2C_3}\frac{1}{\|\mathcal{C}[G;\Gamma]^{-1}\|_{\mathcal{L}(L^2(\Gamma))}} n^{-\gamma-\beta}.$$

Using Lemma 2.1 $\mathcal{C}_n[G_\xi;\Gamma]$ is invertible, and we bound

$$\|\mathcal{C}_n[G_\xi;\Gamma]^{-1} - \mathcal{C}_n[G;\Gamma]^{-1}\|_{\mathcal{L}(Y_n,X_n)} \leq 2C_2 n^{2\beta+\gamma}\|\mathcal{C}[G;\Gamma]^{-1}\|^2_{\mathcal{L}(L^2(\Gamma))}E(\xi). \tag{4.4}$$

Importantly, the quantity on the left tends to zero as $\xi \to \infty$. We use a triangle inequality

$$\|u_n - u_{n,\xi}\|_{L^2(\Gamma)} \leq \|(\mathcal{C}_n[G_\xi;\Gamma]^{-1} - \mathcal{C}_n[G;\Gamma]^{-1})\mathcal{I}_n(G - I)\|_{L^2(\Gamma)} + \|\mathcal{C}_n[G;\Gamma]^{-1}\mathcal{I}_n(G - G_\xi)\|_{L^2(\Gamma)}.$$

Since we have assumed that Γ is bounded and that the norm of $\mathcal{I}_n : C(\Gamma) \to L^2(\Gamma)$ is uniformly bounded in n, we obtain L^2 convergence of u_n to $u_{n,\xi}$ as $\xi \to \infty$:

$$\|u_n - u_{n,\xi}\|_{L^2(\Gamma)} \leq C_3 n^{2\beta+\gamma} E(\xi) \|G - I\|_{L^\infty(\Gamma)} + C_4 n^\beta \|G - G_\xi\|_{L^\infty(\Gamma)} \leq C_5 n^{2\beta+\gamma} E(\xi). \tag{4.5}$$

This proves the three required properties. ■

Remark 4.6. *A good way to interpret this result is to see $E(\xi)$ as the difference in norm between the associated infinite-dimensional operator which is proportional to the uniform difference in the jump matrices. Then* (4.4) *gives the resulting error between the finite-dimensional operators. It is worthwhile to note that if $\alpha = \beta = \gamma = 0$ then δ can be chosen independent of n.*

Now we have the tools needed to address the convergence of the solver. We introduce some notation to simplify matters. At stage j in the solver we solve a SIE on Ω_j. On this domain we need to compare two RHPs:

- $[\tilde{H}_j; \Omega_j]$ and
- $[\tilde{H}_{j,\boldsymbol{n}_j}; \Omega_j]$.

Let U_j be the exact solution of this SIE which is obtained from $[\tilde{H}_j; \Omega_j]$. As an intermediate step we need to consider the numerical solution of $[\tilde{H}_j; \Omega_j]$. We use U_{j,n_j} to denote the numerical approximation of U_j of order n_j. Also, $U_{j,\boldsymbol{n}_j}$ will be used to denote the numerical approximation of the solution of the SIE associated with $[\tilde{H}_{j,\boldsymbol{n}_j}; \Omega_j]$.

Theorem 4.2. *Assume that each problem in Algorithm 4.1 is solvable and k-regular for sufficiently large k. Then the algorithm converges to the true solution of the RHP. More precisely, there exists $\boldsymbol{N}_i$ such that for $\boldsymbol{n}_i > \boldsymbol{N}_i$ we have*

$$\|U_{i,\boldsymbol{n}_i} - U_i\|_{L^2(\Omega_i)} \leq C_k \left[(\max \boldsymbol{n}_i)^{\alpha+\beta} + (\max \boldsymbol{n}_i)^{2\alpha+\gamma} \right]^i \max_{j \leq i} \|\mathcal{I}_n U_j - U_j\|_{H^1(\Omega_j)},$$

where $\mathcal{I}_n$ is the appropriate projection for Ω_j.

Proof: We prove this by induction. Since $U_{1,\boldsymbol{n}_1} = U_{1,n_1}$ the claim follows from Theorem 4.1 for $i = 1$. Now assume the claim is true for all $j < i$. We use Lemma 4.1 to show it is true for i. Using the triangle inequality we have

$$\|U_{i,\boldsymbol{n}_i} - U_i\|_{L^2(\Omega_i)} \leq \|U_{i,\boldsymbol{n}_i} - U_{i,n_i}\|_{L^2(\Omega_i)} + \|U_i - U_{i,n_i}\|_{L^2(\Omega_i)}.$$

Using Theorem 4.1, we bound the second term:

$$\|U_i - U_{i,n_i}\|_{L^2(\Omega_i)} \leq C n_i^{\alpha+\beta} \|\mathcal{I}_n U_i - U_i\|_{H^1(\Omega_i)}.$$

To bound the first term we use (4.5),

$$\|U_{i,\boldsymbol{n}_i} - U_{i,n_i}\|_{L^2(\Omega_i)} \leq C n_i^{2\beta+\gamma} E(\boldsymbol{n}_{i-1}). \tag{4.6}$$

$E(\boldsymbol{n}_{i-1})$ is proportional to the uniform difference of $\tilde{H}_i$ and its approximation obtained through the numerical method, $\tilde{H}_{i,\boldsymbol{n}_{i-1}}$. By the induction hypothesis, if k is sufficiently large, Lemma 4.1, tells us that this difference tends to zero as $\boldsymbol{n}_{i-1} \to \infty$, and the use of (4.5) is justified. More precisely, the Cauchy–Schwartz inequality for each Ω_j, $j < i$ and repeated triangle inequalities results in

$$\|\tilde{H}_i - \tilde{H}_{i,\boldsymbol{n}_{i-1}}\|_{L^\infty(\Omega_i)} \leq C \sum_{j=1}^{i-1} \|U_j - U_{j,n_j}\|_{L^2(\Omega_j)}. \tag{4.7}$$

Combining (4.6) and (4.7) we complete the proof. ■

Remark 4.7. *The requirement that k is large can be made more precise using Definition 4.2 with $m = \max\{l(2\alpha + \gamma), l(\alpha + \beta)\}$. There is little restriction if $(\alpha, \beta, \gamma) = (0, 0, 0)$.*

5 Uniform Approximation

In this section we describe how the above results can be used. First, we briefly describe how to obtain an explicit asymptotic approximation. Second, we use the same ideas to explain how numerics can be used to provide asymptotic approximations. The idea we continue to exploit is that the set of invertible operators between Banach spaces is open. Before we proceed, we define two types of uniform approximation. Let $\{U_n^\xi\}_{\xi \geq 0}$ be a sequence, depending on the parameter ξ, in a Banach space such that for each ξ, $\|U_n^\xi - U^\xi\| \to 0$ as $n \to \infty$ for some U^ξ.

Definition 5.1. *We say the sequence $\{U_n^\xi\}_{\xi \geq 0}$ is* weakly uniform *if for every $\epsilon > 0$ there exists a function $N(\xi) : \mathbb{R}^+ \to \mathbb{N}$ taking finitely many values such that*

$$\|U_{N(\xi)}^\xi - U^\xi\| < \epsilon.$$

Definition 5.2. *We say the sequence $\{U_n^\xi\}_{\xi \geq 0}$ is* strongly uniform *(or just* uniform*) if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N$*

$$\|U_n^\xi - U^\xi\| < \epsilon.$$

The necessity for the definition of a weakly uniform sequence is mostly a technical detail, as we do not see it arise in practice. To illustrate how it can arise we give an example.

Example 5.1. *Consider the sequence*

$$\{U_n^\xi\}_{n,\xi \geq 0} = \left\{ \sin \xi + e^{-n^2} + e^{-(\xi - n)^2} \right\}_{n, \xi \geq 0}.$$

For fixed ξ, $U_n^\xi \to \sin \xi$. We want, for $\epsilon > 0$, while keeping n bounded

$$|U_n^\xi - \sin \xi| = |e^{-n^2} + e^{-(\xi - n)^2}| < \epsilon.$$

We choose $n > \xi$ or if ξ is large enough we choose $0 < n < \xi$. To maintain error that is uniformly less then ϵ we cannot choose a fixed n; it must vary with respect to ξ. When relating to RHPs the switch from $n > \xi$ to $0 < n < \xi$ is related to transitioning into the asymptotic regime.

5.1 Nonlinear Steepest Descent

For simplicity, assume the RHP $[G^\xi, \Gamma^\xi]$ depends on a single parameter $\xi \geq 0$. Further, assume $\Gamma^\xi = \alpha(\xi)\Omega + \beta(\xi)$ where Ω is a fixed contour. It follows that the matrix-valued functions

$$H^\xi(k) = G^\xi(\alpha(\xi)\Omega + \beta(\xi)),$$

are defined on Ω. Assume there exists a matrix H such that $\|H - H^\xi\|_{L^\infty(\Omega)} \to 0$ as $\xi \to \infty$. Use Φ^ξ and Φ to denote the solutions of $[H^\xi; \Omega]$ and $[H; \Omega]$, respectively. Applying Theorem 3.2, Φ^ξ approximates Φ in the limit. If everything works out, we expect to find

$$\Phi^\xi = \Phi + \mathcal{O}(\xi^{-\nu}), \quad \nu > 0. \tag{5.1}$$

Furthermore, if we find an explicit solution to $[H; \Omega]$ we can find an explicit asymptotic formula for the RHP $[G^\xi; \Gamma^\xi]$. This function Φ that solves $[H^\xi; \Omega]$ asymptotically will be referred to as a *parametrix*.

We return to Example 4.1 and introduce a parameter into the problem to demonstrate a situation in which the RHPs on each disjoint contour decouple from each other.

Example 5.2. *Assume $\Gamma^\xi = \Gamma_1^\xi \cup \Gamma_2^\xi$ and*

$$\begin{aligned} \Gamma_1^\xi &= \alpha_1(\xi)\Omega_1 + \beta_1, \\ \Gamma_2^\xi &= \alpha_2(\xi)\Omega_2 + \beta_2, \quad |\beta_1 - \beta_2| > 0. \end{aligned}$$

Assume that each Ω_i is bounded. We consider the L^2 norm of the Cauchy operator applied to a function defined on Γ_1^ξ and evaluated on Γ_2^ξ: $\mathcal{C}_{\Gamma_1^\xi} u(z)|_{\Gamma_2^\xi}$. Explicitly,

$$\mathcal{C}_{\Gamma_1^\xi} u(z) = \frac{1}{2\pi i} \int_{\Gamma_1^\xi} \frac{u(s)}{s-z} dx.$$

This is a Hilbert–Schmidt operator, and

$$\|\mathcal{C}_{\Gamma_1^\xi}\|^2_{\mathcal{L}(L^2(\Gamma_1^\xi), L^2(\Gamma_2^\xi))} \leq \int_{\Gamma_1^\xi} \int_{\Gamma_2^\xi} \frac{|dxdk|}{|x-k|^2}. \tag{5.2}$$

A simple change of variables shows that

$$\|\mathcal{C}_{\Gamma_1^\xi}\|^2_{\mathcal{L}(L^2(\Gamma_1^\xi), L^2(\Gamma_2^\xi))} \leq |\alpha_1(\xi)\alpha_2(\xi)| \int_{\Omega_1} \int_{\Omega_2} \frac{|dsdy|}{|\alpha_1(\xi)s - \alpha_2(\xi)y + \beta_1 - \beta_2|^2}. \tag{5.3}$$

Since the denominator in the integral in (5.3) is bounded away from zero and both Ω_i are bounded, the right-hand side tends to zero if either α_1 or α_2 tend to zero.

This argument, with more contours, can be used to further justify Algorithm 4.1 in this limit by noting that this type of Cauchy operator (evaluation off the contour of integration) constitutes the operator $\mathcal{T}$ in §4. We have the representation

$$\mathcal{C}[G^\xi; \Gamma^\xi] = \begin{bmatrix} \mathcal{C}[G_1^\xi; \Gamma_1^\xi] & 0 \\ 0 & \mathcal{C}[G_2^\xi; \Gamma_2^\xi] \end{bmatrix} + \mathcal{T}^\xi,$$

where $\|\mathcal{T}^\xi\|_{\mathcal{L}(L^2(\Gamma))} \to 0$ as $\xi \to \infty$.

This analysis follows similarly in some cases when β_i depends on ξ. For example, when

$$\inf_{t \in S} |\beta_1(\xi) - \beta_2(\xi)| = \delta > 0. \tag{5.4}$$

One can extend this to the case where (5.4) is not bounded away from zero but approaches zero slower than $a_1(\xi)a_2(\xi)$. For simplicity we just prove results for β_i being constant.

Furthermore, the norms of the inverses are related. When each of the $\alpha_i(\xi)$ are sufficiently small, there exists $C > 1$ such that

$$\frac{1}{C} \|\mathcal{C}[G; \Gamma]^{-1}\|_{\mathcal{L}(L^2(\Gamma))} \leq \max_i \|\mathcal{C}[G_i; \Gamma_i]^{-1}\|_{\mathcal{L}(L^2(\Gamma_i))} \leq C \|\mathcal{C}[G; \Gamma]^{-1}\|_{\mathcal{L}(L^2(\Gamma))}. \tag{5.5}$$

Due to the simplicity of the scalings we allow the norms of the operators $\mathcal{C}[G_i; \Gamma_i]$ and $\mathcal{C}[G_i; \Gamma_i]^{-1}$ to coincide with their scaled counterparts $\mathcal{C}[H_i; \Omega_i]$ and $\mathcal{C}[H_i; \Omega_i]^{-1}$.

The choice of these scaling parameters is not a trivial task. We use the following rule of thumb:

Assumption 5.1. *If the jump matrix G has a factor $e^{\xi\theta}$ and β_k corresponds to a qth order stationary point β_k (*i.e.*, $\theta(z) \backsim C(z - \beta_k)^q$), then the scaling which achieves asymptotic stability is $\alpha_k(\xi) = |\xi|^{-1/q}$.*

We prove the validity of this assumption for the deformations below on a case-by-case basis. We need one final result to guide deformations. We start with a RHP posed on an unbounded and connected contour. In all cases we need to justify the truncation of this contour, hopefully turning it into disconnected contours.

Lemma 5.1. *Assume $[G; \Gamma]$ is k-regular. For every $\epsilon > 0$ there exists a function G_ϵ defined on Γ and a bounded contour $\Gamma_\epsilon \subset \Gamma$ such that:*

- $G_\epsilon = I$ *on* $\Gamma \setminus \Gamma_\epsilon$,
- $\|G_\epsilon - G\|_{W^{k,\infty}(\Gamma) \cap H^k(\Gamma)} < \epsilon$

- $[G_\epsilon, \Gamma_\epsilon]$ *is k-regular and*

$$\|\mathcal{C}[G;\Gamma] - \mathcal{C}[G_\epsilon, \Gamma]\|_{\mathcal{L}(L^2(\Gamma))} < \epsilon \|\mathcal{C}_\Gamma^-\|_{\mathcal{L}(L^2(\Gamma))}.$$

Proof: A matrix-valued function f is chosen such that

- $f|_{\Gamma_i} \in C^\infty(\Gamma_i)$,
- $f = I$ in a neighborhood of all intersection points,
- f has compact support, and
- $\|(G-I)f + I - G\|_{W^{k,\infty}(\Gamma)\cap H^k(\Gamma)} < \epsilon$.

Equate $G_\epsilon = (G-I)f + I$. The last property follows immediately. ■

This justifies the truncation of infinite contours to finite ones and it shows this process preserves smoothness. When it comes to numerical computations, we truncate contours when the jump matrix is, to machine epsilon, the identity matrix. In what follows we assume this truncation is performed and we ignore the error induced.

5.2 Direct Estimates

As before, we are assuming we have a RHP that depends on a parameter ξ, $[G^\xi; \Gamma^\xi]$ and Γ^ξ is bounded. Here we use *a priori* bounds on the solution of the associated SIE which are uniform in ξ to prove the uniform approximation. In general, when this is possible, it is the simplest way to proceed.

Our main tool is Corollary 3.1. We can easily estimate the regularity of the solution of each problem $[H_i; \Omega_i]$ provided we have some information about $\|\mathcal{C}[G_i^\xi; \Gamma_i^\xi]^{-1}\|_{\mathcal{L}(L^2(\Gamma_i))}$ or equivalently $\|\mathcal{C}[H_i^\xi; \Omega_i]^{-1}\|_{\mathcal{L}(L^2(\Omega_i))}$. We address how to estimate this later in this section. First, we need a statement about how regularity is preserved throughout Algorithm 4.1. Specifically, we use information from the scaled jumps H_i and the local inverses $\mathcal{C}[H_i; \Omega_i]^{-1}$ to estimate global regularity. The following theorem uses this to prove strong uniformity of the numerical method.

Theorem 5.1. *Assume*

- $\{[G^\xi, \Gamma^\xi]\}_{\xi \geq 0}$ *is a sequence of k-regular RHPs,*
- *the norm of* $\mathcal{C}[H_i^\xi, \Omega_i]^{-1}$ *is uniformly bounded in* ξ*,*
- $\|H_i^\xi\|_{W^{k,\infty}(\Omega_i)} \leq C$*, and*
- $\alpha_i(\xi) \to 0$ *as* $\xi \to \infty$*.*

Then if k and ξ *are sufficiently large*

- *Algorithm 4.1 applied to* $\{[G^\xi, \Gamma^\xi]\}_{\xi \geq 0}$ *has solutions at each stage,*
- $\|U_j^\xi\|_{H^k(\Omega_i)} \leq P_k$ *where* P_k *depends on* $\|H_i^\xi\|_{H^k(\Omega_i)\cap W^{k,\infty}(\Omega_i)}$, $\|\mathcal{C}[H_i^\xi; \Omega_i]^{-1}\|_{\mathcal{L}(L^2(\Omega_i))}$ *and* $\|U_j^\xi\|_{L^2(\Omega_j)}$ *for* $j < i$ *and*
- *the approximation* $U_{i,\boldsymbol{n_i}}^\xi$ *of* U_i^ξ *(the solution of the SIE) at each step in Algorithm 4.1 converges uniformly in* ξ *as* $\boldsymbol{n}_i \to \infty$*, that is, convergence is strongly uniform.*

Proof: First, we note that since $\alpha_i(\xi) \to 0$, (5.3) shows that jump matrix $\tilde{H}_i^\xi$ for the RHP solved at stage i in Algorithm 4.1 tends uniformly to H_i^ξ. This implies the solvability of the RHPs at each stage in Algorithm 4.1, and the bound

$$\|\mathcal{C}[\tilde{H}_i^\xi; \Omega_i]^{-1}\|_{\mathcal{L}(L^2(\Gamma))} \leq C \|\mathcal{C}[H_i^\xi; \Omega_i]^{-1}\|_{\mathcal{L}(L^2(\Gamma))},$$

for sufficiently large ξ. As before, C can be taken to be independent of ξ. We claim that $\|U_i^\xi\|_{H^k(\Omega_i)}$ is uniformly bounded. We prove this by induction. When $i = 1$, $U_1^\xi = \mathcal{C}[H_i^\xi, \Omega_i]^{-1}(H_i^\xi - I)$ and the claim follows from Corollary 3.1. Now assume the claim is true for $j < i$. All derivatives of the jump matrix $\tilde{H}_i^\xi$ depend on the Cauchy transform of U_j^ξ evaluated away from Ω_j and H_i^ξ. The former is bounded by the induction hypothesis and the later is bounded by assumption. Again, sing Corollary 3.1 we obtain the uniform boundedness of $\|U_i^\xi\|_{H^k(\Omega_i)}$. Theorem 4.2 implies that convergence is uniform in ξ. ■

The most difficult part about verifying the hypotheses of this theorem is establishing an estimate of $\|\mathcal{C}[H_i^\xi; \Omega_i^\xi]^{-1}\|_{\mathcal{L}(L^2(\Omega_i))}$ as a function of ξ. A very useful fact is that once the solution Ψ^ξ of the RHP $[G^\xi; \Gamma^\xi]$ is known then the inverse of the operator is also known:

$$\mathcal{C}[G^\xi; \Gamma^\xi]^{-1}u = \mathcal{C}^+_{\Gamma^\xi}[u(\Psi^\xi)^+][(\Psi^\xi)^{-1}]^+ - \mathcal{C}^-_{\Gamma^\xi}[u(\Psi^\xi)^+][(\Psi^\xi)^{-1}]^-. \tag{5.6}$$

This is verified by direct computation. When Ψ^ξ is known approximately, *i.e.*, when a parametrix is known, then estimates on the boundedness of the inverse can be reduced to studying the L^∞ norm of the parametrix. Then (5.5) can be used to relate this to each $\mathcal{C}[G_i^\xi; \Gamma_i^\xi]^{-1}$ which gives an estimate for the norm of $\mathcal{C}[H_i^\xi; \Omega_i]^{-1}$. We study this further in section 7.

5.3 Failure of Direct Estimates

We study a toy RHP to motivate where the direct estimates can fail. Let $\phi(x)$ be a smooth function with compact support in $(-1, 1)$ satisfying $\max_{[-1,1]} |\phi(x)| = 1/2$. Consider the following scalar RHP for a function μ:

$$\mu^+(x) = \mu^-(x)(1 + \phi(x)(1 + \xi^{-1/2}e^{i\xi x})), \tag{5.7}$$
$$\mu(\infty) = 1, \quad x \in [-1, 1], \quad \xi > 0. \tag{5.8}$$

This problem can be solved explicitly but we study it from the linear operator perspective instead. From the boundedness assumption on ϕ, a Neumann series argument gives the invertiblity of the singular integral operator and uniform boundedness of the L^2 inverse in ξ. Using the estimates in Corollary 3.1 we obtain useless bounds, which all grow with ξ. Intuitively, the solution to (5.7) is close, in L^2 to the solution to

$$\nu^+(x) = \nu^-(x)(1 + \phi(x)), \tag{5.9}$$
$$\nu(\infty) = 1, \quad x \in [-1, 1], \quad \xi > 0, \tag{5.10}$$

which trivially has uniform bounds on its Sobolev norms. In the next section we introduce the idea of a numerical parametrix which helps resolve this complication.

5.4 Extension to Indirect Estimates

In this section we assume minimal hypotheses for dependence of the sequence $\{[G^\xi; \Gamma^\xi]\}_{\xi \geq 0}$ on ξ. Specifically we require only that the map $\xi \mapsto H_i^\xi$ is continuous from $\mathbb{R}^+$ to $L^\infty(\Omega_i)$ for each i. We do not want to hypothesize more as that would alter the connection to the method of nonlinear steepest descent which only requires uniform convergence of the jump matrix. In specific cases, stronger results can be obtained by requiring the map $\xi \mapsto H_i^\xi$ to be continuous from $\mathbb{R}^+$ to $W^{k,\infty}(\Omega_i)$.

The fundamental result we need to prove a uniform approximation theorem is the continuity of Algorithm 4.1 with respect to uniform perturbations in the jump matrix. With the jump matrix G we associated H_j, the scaled restriction of G to Γ_j. With G we also associated U_j, the solution of the SIE obtained from $[\tilde{H}_j; \Omega_j]$. In what follows we will have another jump matrix J and analogously we use K_j to denote the scaled restriction of J and P_j to denote the solution of the SIE obtained from $[\tilde{K}_j, \Omega_j]$.

Lemma 5.2. *Assume $\{[G^\xi, \Gamma^\xi]\}_{\xi \geq 0}$ is a sequence of 1-regular RHPs such that $\xi \mapsto H_i^\xi$ is continuous from $\mathbb{R}^+$ to $L^\infty(\Omega_i)$ for each i. Then for sufficiently large, but fixed, $\boldsymbol{n}_i$ the map $\xi \mapsto U_{i,\boldsymbol{n}_i}^\xi$ is continuous from $\mathbb{R}^+$ to $L^2(\Omega_i)$ for each i.*

Proof: We prove this by induction on i. For $i = 1$ the claim follows from Lemma 4.1. Now assume the claim is true for $j < i$ and we prove it holds for i. We show the map is continuous at η for $\eta \geq 0$. First, from Lemma 4.1

$$\|U^{\eta}_{i,\boldsymbol{n}_i} - U^{\xi}_{i,\boldsymbol{n}_i}\|_{L^2(\Omega)} \leq C n_i^{2\alpha+\gamma} E(\xi, \eta),$$

where $E(\xi, \eta)$ is proportional to $\|\tilde{H}^{\eta}_{i,\boldsymbol{n}_{i-1}} - \tilde{H}^{\xi}_{i,\boldsymbol{n}_{i-1}}\|_{L^\infty(\Omega_i)}$. A similar argument as in Theorem 4.2 gives

$$\|\tilde{H}^{\eta}_{i,\boldsymbol{n}_{i-1}} - \tilde{H}^{\xi}_{i,\boldsymbol{n}_{i-1}}\|_{L^\infty(\Omega_i)} \leq C(\eta, \boldsymbol{n}_i) \sum_{j=1}^{i-1} \|U^{\eta}_{j,\boldsymbol{n}_j} - U^{\xi}_{j,\boldsymbol{n}_j}\|_{L^2(\Omega_j)},$$

for $|\xi - \eta|$ sufficiently small. By assumption, the right-hand side tends to zero as $\xi \to \eta$ and this proves the lemma. ■

It is worthwhile noting that the arguments in Lemma 5.2 show the same continuity for the infinite-dimensional, non-discretized problem. Now we show weak uniform convergence of the numerical scheme on compact sets.

Proposition 5.1. *Assume $\{[G^\xi, \Gamma^\xi]\}_{\xi \geq 0}$ is a sequence of k-regular RHPs such that all the operators in Algorithm 4.1 are invertible for every ξ. Assume that k is sufficiently large so that the approximations from Algorithm 4.1 converge for every $\xi \geq 0$. Then there exists a vector-valued function $\boldsymbol{N}(i, \xi)$ that takes finitely many values such that*

$$\|U^{\xi}_{i,\boldsymbol{N}(i,\xi)} - U^{\xi}_i\|_{L^2(\Omega_i)} < \epsilon.$$

Moreover if the numerical method is of type $(0, 0, 0)$ then convergence is strongly uniform.

Proof: Let $S \subset \mathbb{R}^+$ be compact. It follows from Lemma 5.2 that the function $E(\xi, \boldsymbol{n}, i) = \|U^{\xi}_{i,\boldsymbol{n}} - U^{\xi}_i\|_{L^2(\Gamma_i)}$ is a continuous function of ξ for fixed $\boldsymbol{n}$. For $\epsilon > 0$ find an $\boldsymbol{n}_\xi$ such that $E(\xi, \boldsymbol{n}_\xi, i) < \epsilon/2$. By continuity, define $\delta_\xi(\boldsymbol{n}_\xi) > 0$ so that $E(s, \boldsymbol{n}_\xi, i) < \epsilon$ for $|s - \xi| < \delta_\xi$. Define the ball $B(\xi, \delta) = \{s \in \mathbb{R}^+ : |s - \xi| < \delta\}$. The open sets $\{B(\xi, \delta_\xi)\}_{\xi \in S}$ cover S and we can select a finite subcover $\{B(\xi_j, \delta_{\xi_j})\}_{j=1}^N$. We have $E(s, \boldsymbol{n}_{\xi_j}, i) < \epsilon$ whenever $s \in B(\xi_j, \delta_{\xi_j})$. To prove the claim for a method of type $(0, 0, 0)$, we use the fact that δ_ξ can be taken independent of $\boldsymbol{n}_\xi$ and that $E(s, \boldsymbol{n}, i) < \epsilon$ for every $\boldsymbol{n} > \boldsymbol{n}_\xi$. ■

Definition 5.3. *Given a sequence of k-regular RHPs $\{[G^\xi, \Gamma^\xi]\}_{\xi \geq 0}$ such that*

- $\Gamma^\xi = \Gamma^\xi_1 \cup \cdots \cup \Gamma^\xi_\ell$, *and*
- $\Gamma^\xi_i = \alpha_i(\xi)\Omega_i + \beta_i$,

another sequence of k-regular RHPs $\{[J^\xi, \Sigma^\xi]\}_{\xi \geq 0}$ is said to be a numerical parametrix if

- $\Sigma^\xi = \Sigma^\xi_1 \cup \cdots \cup \Sigma^\xi_\ell$,
- $\Sigma^\xi_i = \gamma_i(\xi)\Omega_i + \sigma_i$,
- *For all i*
$$J^\xi(\gamma_i(\xi)k + \sigma_i) - G^\xi(\alpha_i(\xi)k + \beta_i) \to 0, \tag{5.11}$$
uniformly on Ω_i as $\xi \to \infty$,
- *the norms of the operators and inverse operators at each step in Algorithm 4.1 are uniformly bounded in ξ, implying uniform boundedness of J^ξ in ξ, and*
- *the approximation $P^{\xi}_{i,\boldsymbol{n_i}}$ of P^ξ_i (the solution of the SIE) at each step in Algorithm 4.1 converges uniformly as $\min \boldsymbol{n}_i \to \infty$.*

This definition hypothesizes desirable conditions on a nearby limit problem for the sequence $\{[G^\xi,\Gamma^\xi]\}_{\xi\geq 0}$. Under the assumption of this nearby limit problem we are able to obtain a uniform approximation for the solution of the original RHP.

Lemma 5.3. *Assume there exists a numerical parametrix $\{J_\xi,\Sigma_\xi\}_{\xi>0}$ for a sequence of RHPs $\{[G_\xi,\Gamma_\xi]\}_{\xi\geq 0}$. Then for every $\epsilon>0$ there exists $\boldsymbol{N}_i$ and $T>0$ such that, at each stage in Algorithm 4.1,*

$$\|U^\xi_{i,\boldsymbol{N}_i}-U^\xi_i\|_{L^2(\Omega_i)}<\epsilon \text{ for } \xi>T. \tag{5.12}$$

Furthermore, if the numerical method is of type $(0,0,0)$, then (5.12) is true with $\boldsymbol{N}_i$ replaced by any $\boldsymbol{M}_i>\boldsymbol{N}_i$.

Proof: At each stage in Algorithm 4.1 we have

$$\|U^\xi_{i,\boldsymbol{n}_i}-U^\xi_i\|_{L^2(\Omega_i)}\leq \|U^\xi_{i,\boldsymbol{n}_i}-P^\xi_{i,\boldsymbol{n}_i}\|_{L^2(\Omega_i)}+\|P^\xi_{i,\boldsymbol{n}_i}-P^\xi_i\|_{L^2(\Omega_i)}+\|P^\xi_i-U^\xi_i\|_{L^2(\Omega)}. \tag{5.13}$$

Since $P^\xi_{i,\boldsymbol{n}_i}$ originates from a numerical parametrix we know that $\|P^\xi_{i,\boldsymbol{n}_i}-P^\xi_i\|_{L^2(\Omega_i)}\to 0$ uniformly in ξ as $\boldsymbol{n}_i$ is increased. Furthermore, $\|P^\xi_i-U^\xi_i\|_{L^2(\Omega)}$ depends only on ξ and tends to zero as $\xi\to\infty$. The main complication comes from the fact that a bound on $\|U^\xi_{i,\boldsymbol{n}_i}-P^\xi_{i,\boldsymbol{n}_i}\|_{L^2(\Omega_i)}$ from (4.4) depends on both $\boldsymbol{n}_{i-1}$ and ξ if the method is not of type $(0,0,0)$. The same arguments as in Lemma 5.2 show this tends to zero. Therefore we choose $\boldsymbol{n}_i$ large enough so that the second term in (5.13) is less than $\epsilon/3$. Next, we choose ξ large enough so that the sum of the remaining terms is less than $2/3\epsilon$. If the method is of type $(0,0,0)$ this sum remains less than ϵ if $\boldsymbol{n}_i$ is replaced with $\boldsymbol{n}$ for $\boldsymbol{n}>\boldsymbol{n}_i$. This proves the claims. ∎

Now we prove the uniform approximation theorem.

Theorem 5.2. *Assume $\{[G^\xi,\Gamma^\xi]\}_{\xi\geq 0}$ is a sequence of k-regular RHPs for k sufficiently large so that Algorithm 4.1 converges for each ξ. Assume there exists a numerical parametrix as $\xi\to\infty$. Then Algorithm 4.1 produces a weakly uniform approximation to the solution of $\{[G^\xi,\Gamma^\xi]\}_{\xi\geq 0}$. Moreover, convergence is strongly uniform if the method is of type $(0,0,0)$.*

Proof: Lemma 5.3 provides an $M>0$ and $\boldsymbol{N}_1(i)$ such that if $\xi>M$ then

$$\|U^\xi_{i,\boldsymbol{N}_1(i)}-U^\xi_i\|_{L^2(\Omega_i)}<\epsilon, \text{ for every } i.$$

According to Theorem 5.1 there is an $\boldsymbol{N}_2(\xi,i)$ such that

$$\|U^\xi_{i,\boldsymbol{N}_2(\xi,i)}-U^\xi_i\|_{L^2(\Omega_i)}<\epsilon, \text{ for every } i.$$

The function

$$\boldsymbol{N}(\xi,i)=\begin{cases}\boldsymbol{N}_1(i), & \text{if } \xi>M,\\ N_2(\xi,i), & \text{if } \xi\leq \boldsymbol{M},\end{cases}$$

satisfies the required properties for weak uniformity. Strong uniformity follows in a similar way from Lemma 5.3 and Theorem 5.1. ∎

Remark 5.1. *This proves weak uniform convergence of the numerical method for the toy problem introduced in §5.3: we can take the RHP for ν as a numerical parametrix.*

The seemingly odd restrictions for the general theorem are a consequence of poorer operator convergence rates when n is large. A well-conditioned numerical method does not suffer from this issue. It is worth noting that using direct estimates is equivalent to requiring that the RHP itself satisfies the properties of a numerical parametrix.

In what follows, we want to show a given sequence of RHPs is a numerical parametrix. The reasoning for the following result is two-fold. First, we hypothesize only conditions which are easily checked in practice. Second, we want to connect the stability of numerical approximation with the use of local, model problems in nonlinear steepest descent.

Proposition 5.2. *Assume*

- $\{[J^\xi, \Sigma^\xi]\}_{\xi \geq 0}$ *is a sequence of k-regular RHPs,*
- $\mathcal{C}[K_i^\xi, \Omega_i]^{-1}$ *has norm uniformly bounded in ξ,*
- $\|K_i^\xi\|_{W^{k,\infty}(\Omega_i)} \leq C$, *and*
- $\gamma_i(\xi) \to 0$ *as $\xi \to \infty$.*

Then if k and ξ are sufficiently large

- *Algorithm 4.1 applied to $\{[J^\xi, \Sigma^\xi]\}_{\xi \geq 0}$ has solutions at each stage and*
- $\{[J^\xi, \Sigma^\xi]\}_{\xi \geq 0}$ *satisfies the last two properties of a numerical parametrix (Definition 5.3).*

Proof: The proof of this is essentially the same as Theorem 5.1 ■
.

Remark 5.2. *Due to the decay of γ_i, the invertiblity of each of $\mathcal{C}[K_i^\xi; \Omega_i]$ is equivalent to that of $\mathcal{C}[G^\xi; \Gamma^\xi]$.*

This proposition states that a numerical parametrix only needs to be locally reliable; we can consider each shrinking contour as a separate RHP as far as the analysis is concerned.

6 A Numerical Realization

In [11], a numerical framework was constructed for computing solutions to RHPs, based on a method used in [12] for computing solutions to the undeformed Painlevé II RHP. This framework is based on Chebyshev interpolants. Consider the RHP $[G; \Gamma]$, $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_\ell$, where each Γ_i is bounded and is a Möbius transformation of the unit interval:

$$M_i([-1,1]) = \Gamma_i.$$

Definition 6.1. *The Chebyshev points of the second kind are*

$$\boldsymbol{x}^{[-1,1],n} = \begin{bmatrix} x_1^{[-1,1],n} \\ \vdots \\ x_n^{[-1,1],n} \end{bmatrix} = \begin{bmatrix} -1 \\ \cos \pi \left(1 - \frac{1}{n-1}\right) \\ \vdots \\ \cos \frac{\pi}{n-1} \\ 1 \end{bmatrix}.$$

The mapped Chebyshev points are denoted

$$\boldsymbol{x}^{i,n} = M_i(\boldsymbol{x}^{[-1,1]}).$$

Given a continuous function f_i defined on Γ_i we can find a unique interpolant at $\boldsymbol{x}^{i,n}$ using mapped Chebyshev polynomials. Given a function, f defined on the whole of Γ, we define $\mathcal{I}_n$ to be this interpolation projection applied to the restriction of f on each Γ_i. Clearly,

$$\mathcal{I}_n : H^1(\Gamma) \to H^1(\Gamma)$$

and, because $\boldsymbol{x}^{j,n}$ contains all junction points,

$$\mathcal{I}_n : H_z^1(\Gamma) \to H_z^1(\Gamma).$$

The framework in [11] is given by the pair $(\mathcal{I}_n, \mathcal{I}_n)$ and the matrix $\mathcal{C}_n[G; \Gamma]$ is equal to $\mathcal{I}_n \mathcal{C}[G; \Gamma] \mathcal{I}_n$ with some unbounded components subtracted; obeying the requirement that the two operators agree on $H_z^1(\Gamma)$.

Now we address the properties required in Definition 4.2.

Lemma 6.1. *When $G \in W^{1,\infty}(\Gamma)$, the numerical method in [11] satisfies:*

- $\mathcal{I}_n$ *is uniformly bounded in n from $C(\Gamma)$ to $L^2(\Gamma)$ when Γ is bounded.*
- $\|\mathcal{C}_n[G;\Gamma]\|_{\mathcal{L}(H^1_z(\Gamma),Y_n)} \leq C(1+\|G-I\|_{L^\infty(\Gamma)}\|\mathcal{C}^-_\Gamma\|_{\mathcal{L}(L^2(\Gamma))})$.
- $\|\mathcal{T}_n[G;\Gamma]\|_{\mathcal{L}(X_n,Y_n)} \leq Cn^2\|G-I\|_{L^\infty(\Gamma)}\|\mathcal{C}^-_\Gamma\|_{\mathcal{L}(L^2(\Gamma))}$.
- $\|\mathcal{I}_n u - u\|_{H^1(\Gamma)} \leq C_s n^{2-s}\|u\|_{H^s(\Gamma)}$.

Proof: First, note that these constants depend on Γ. Using the Dirichlet kernel one proves that $\mathcal{I}_n$ is uniformly bounded from $C(\Gamma)$ to an L^2 space with the Chebyshev weight [2]. The norm on this weighted space dominates the usual $L^2(\Gamma)$ norm, proving the first result. For the second statement we take $u \in H^1_z(\Gamma)$ and consider

$$\|\mathcal{I}_n - \mathcal{I}_n(G-I)\mathcal{C}^-_\Gamma u\|_{L^2(\Gamma)} \leq \|\mathcal{I}_n\|_{\mathcal{L}(C(\Gamma),L^2(\Gamma))}(1+\|G-I\|_{L^\infty(\Gamma)}\|\mathcal{C}^-_\Gamma\|_{\mathcal{L}(H^1_z(\Gamma),H^1(\Gamma))}\|u\|_{H^1(\Gamma)}).$$

Since Y_n is equipped with the $L^2(\Gamma)$ norm and $\|\mathcal{C}^-_\Gamma\|_{\mathcal{L}(H^1_z(\Gamma),H^1(\Gamma))} = \|\mathcal{C}^-_\Gamma\|_{\mathcal{L}(L^2(\Gamma))}$ we obtain the second property. We then use for $u \in X_n$ (see §4) that $\|u\|_{H^1(\Gamma)} \leq Cn^2\|u\|_{L^2(\Gamma)}$ to obtain

$$\|\mathcal{T}_n[G;\Gamma]\|_{\mathcal{L}(X_n,Y_n)} \leq Cn^2\|G-I\|_{L^\infty(\Gamma)}\|\mathcal{C}^-_\Gamma\|_{\mathcal{L}(L^2(\Gamma))}.$$

The last statement follows from estimates in [14] for the pseudo-spectral derivative. ■

The final property we need to obtain an admissible numerical method, the boundedness of the inverse, is a very difficult problem. We can verify, *a posteriori*, that the norm of the inverse does not grow too much. In general, for this method, we see at most logarithmic growth. We make the following assumption.

Assumption 6.1. *For the framework in [11] we assume that whenever $[G;\Gamma]$ is 1-regular and $\mathcal{C}[G;\Gamma]^{-1}$ exists on $L^2(\Gamma)$ as a bounded operator we have for $n > N$*

$$\|\mathcal{C}_n[G;\Gamma]^{-1}\|_{\mathcal{L}(Y_n,X_n)} \leq Cn^\beta\|\mathcal{C}[G;\Gamma]^{-1}\|_{\mathcal{L}(L^2(\Gamma))}, \quad \beta > 0. \tag{6.1}$$

Remark 6.1. *This assumption is known to be true for a similar collocation method on the unit circle using Laurent monomials [13].*

With this assumption the numerical method associated with $(\mathcal{I}_n, \mathcal{I}_n)$ is of type $(0,\beta,2)$. In light of Theorem 4.1, we expect spectral convergence and the bound in Assumption 6.1 does not prevent convergence. We combine Assumption 6.1, Theorem 4.1 and Theorem 3.1 to obtain

$$\|u-u_n\|_{L^2(\Gamma)} \leq C(\|\mathcal{C}[G;\Gamma]^{-1}\|_{\mathcal{L}(L^2(\Gamma))}(1+\|G-I\|_{L^\infty(\Gamma)}\|\mathcal{C}^-_\Gamma\|_{\mathcal{L}(L^2(\Gamma))})n^{2+\beta-k}\|u\|_{H^k(\Gamma)}. \tag{6.2}$$

7 Application to Painlevé II

We present the RHP for the solution of the Painlevé II ODE (1.1). Let $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_6$ with $\Gamma_i = \{se^{i\pi(i/3-1/6)} : s \in \mathbb{R}^+\}$, *i.e.*, Γ consists of six rays emanating from the origin, see Figure 2. The jump matrix is defined by $G(\lambda) = G_i(\lambda)$ for $\lambda \in \Gamma_i$, where

$$G_i(x;\lambda) = G_i(\lambda) = \begin{cases} \begin{bmatrix} 1 & s_i e^{-i8/3\lambda^3-2ix\lambda} \\ 0 & 1 \end{bmatrix} & \text{if } i \text{ is even,} \\ \begin{bmatrix} 1 & 0 \\ s_i e^{i8/3\lambda^3+2ix\lambda} & 1 \end{bmatrix} & \text{if } i \text{ is odd.} \end{cases}$$

From the solution Φ of $[G;\Gamma]$, the Painlevé function is recovered by the formula

$$u(x) = \lim_{z\to\infty} z\Phi_{12}(z),$$

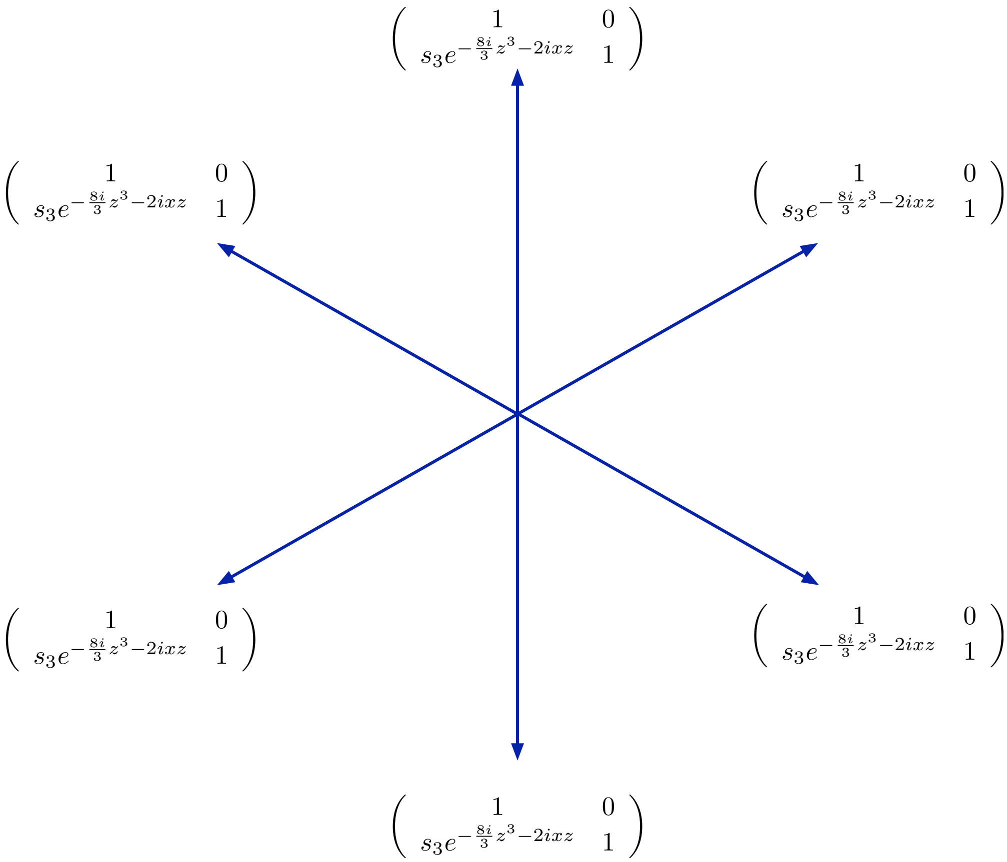

Figure 2: The contour and jump matrix for the Painlevé II RHP.

where the subscripts denote the $(1,2)$ entry. This RHP was solved numerically in [12].

For large $|x|$, the jump matrices G are increasingly oscillatory. We combat this issue by deforming the contour so that these oscillations turn to exponential decay. To simplify this procedure, and to start to mold the RHP into the abstract form in §5, we first rescale the RHP. If we let $z = \sqrt{|x|}\lambda$, then the jump contour Γ is unchanged, and

$$\Phi^+(z) = \Phi^+(\sqrt{|x|}\lambda) = \Phi^-(\sqrt{|x|}\lambda)G(\sqrt{|x|}\lambda) = \Phi^-(z)G(z),$$

where $G(z) = G_i(z)$ on Γ_i with

$$G_i(z) = \begin{cases} \begin{bmatrix} 1 & s_i e^{-\xi\theta(z)} \\ 0 & 1 \end{bmatrix} & \text{if } i \text{ is even,} \\ \begin{bmatrix} 1 & 0 \\ s_i e^{\xi\theta(z)} & 1 \end{bmatrix} & \text{if } i \text{ is odd,} \end{cases}$$

$\xi = |x|^{3/2}$ and

$$\theta(z) = \frac{2i}{3}\left(4z^3 + 2e^{i\arg x}z\right).$$

Then

$$u(x) = \lim_{\lambda\to\infty} \lambda\Phi_{12}(x;\lambda) = \sqrt{x}\lim_{\lambda\to\infty} z\Phi_{12}(x;z). \tag{7.1}$$

We assume that $s_1 s_3 > 1$ and $x < 0$. We deform Γ to pass through the stationary points $\pm 1/2$, resulting in the RHP on the left of Figure 4.

The function

$$G_6G_1G_2 = \begin{bmatrix} 1 - s_1s_3 & s_1e^{-\xi\theta} \\ s_2e^{\xi\theta} & 1 + s_1s_2 \end{bmatrix},$$

has terms with $\exp(\pm\xi\theta)$. It cannot decay to the identity when deformed in the complex plane. We can resolve this issue by using *lensing* [4]. Suppose we factor the jump function as ABC. It is possible to separate the single contour into three contours, as in Figure 3, assuming A and C are analytic between the original contour and continuous up to the new contour. If $\tilde{\Phi}$ satisfies the jumps on the split contour, it is clear that we can recover Φ by defining $\Phi = \tilde{\Phi}C$ between the top contour and the original contour, $\Phi = \tilde{\Phi}A^{-1}$ between the original contour and the bottom contour, and $\Phi = \tilde{\Phi}$ everywhere else, see the bottom of Figure 3. As in the standard deformation of contours, the limit at infinity is unchanged.

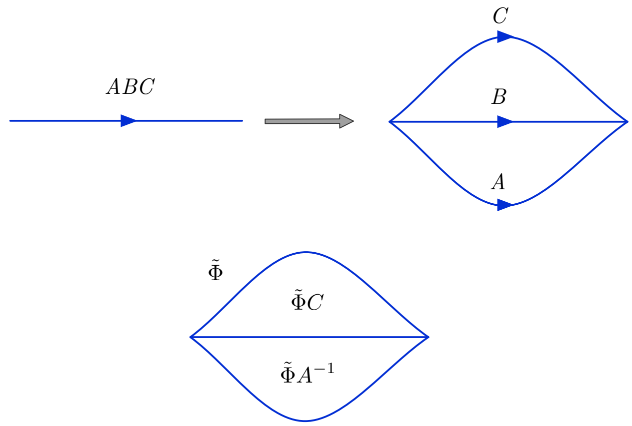

Figure 3: Demonstration of the lensing process.

Now consider the LDU factorization:

$$G_6G_1G_2 = LDU = \begin{bmatrix} 1 & 0 \\ e^{-\zeta\theta}\frac{s_1}{1-s_1s_3} & 1 \end{bmatrix} \begin{bmatrix} 1 - s_1s_3 & 0 \\ 0 & \frac{1}{1-s_1s_3} \end{bmatrix} \begin{bmatrix} 1 & e^{\zeta\theta}\frac{s_1}{1-s_1s_3} \\ 0 & 1 \end{bmatrix}.$$

U decays near $i\infty$, L decays near $-i\infty$, both go to to the identity matrix at infinity and D is constant. Moreover, the oscillators on L and U are precisely those of the original G matrices. Therefore, we reuse the path of steepest descent, and obtain the deformation on the right of Figure 4. The LDU decomposition is

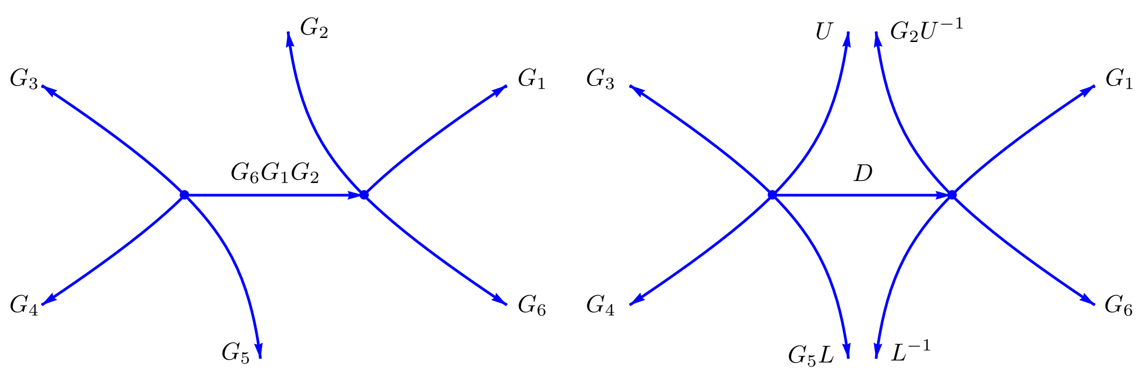

Figure 4: Left: Initial deformation along the paths of steepest descent. Right: The deformed contour after lensing.

valid under the assumption $s_1s_2 > 1$.

7.1 Removing the connected contour

Although the jump matrix D is non-oscillatory (in fact, constant), it is still incompatible with the theory presented in §5: we need the jump matrix to approach the identity matrix away from the stationary points. Therefore, it is necessary to remove this connecting contour. Since $D = \operatorname{diag}(d_1, d_2)$ is diagonal, we can solve $P^+ = P^- D$ with $P(\infty) = I$ on $(-1/2, 1/2)$ in closed form [7]:

$$P(z) = \begin{bmatrix} \left(\frac{2x+1}{2x-1}\right)^{i \log d_1/2\pi} & 0 \\ 0 & \left(\frac{2x+1}{2x-1}\right)^{i \log d_2/2\pi} \end{bmatrix}.$$

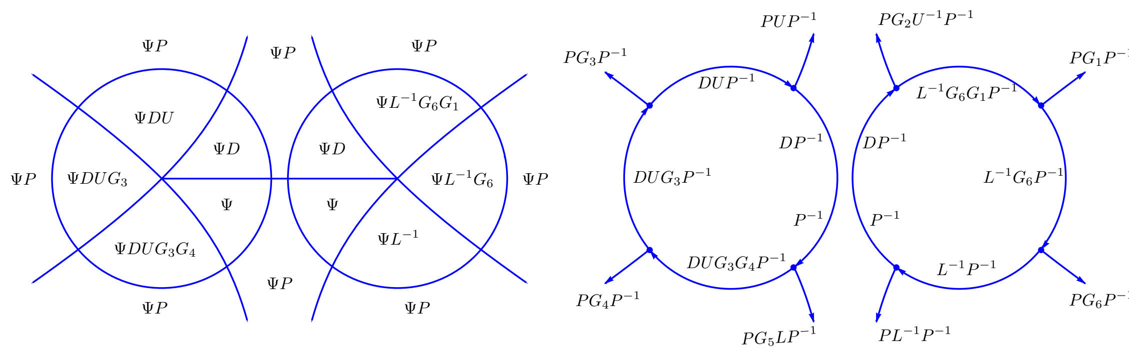

Figure 5: Left: Definition of Φ in terms of Ψ. Right: Jump contour for Ψ.

This parametrix solves the desired RHP for any choice of branch of the logarithm. However, we must choose the branch so that the singularity is square integrable [7]. In this case this is accomplished by choosing the standard choice of branch.

We write

$$\Phi = \Psi P.$$

Since P satisfies the required jump on $(-1/2, 1/2)$, Ψ has no jump there. Moreover, on each of the remaining curves we have

$$\Psi^+ = \Phi^+ P^{-1} = \Phi^- G P^{-1} = \Psi^{-1} P G P^{-1},$$

and our jump matrix becomes PGP^{-1}. Unfortunately, we have introduced singularities at $\pm 1/2$ and the theory of §5 requires some smoothness of the jump matrix. This motivates alternate definitions for Ψ in circles around the stationary points. In particular, we define Φ in terms of Ψ by the left panel of Figure 5, where Ψ has the jump matrix defined in the right. A quick check demonstrates that this definition of Φ indeed satisfies the required jump relations.

We are ready to apply Algorithm 4.1. Define

$$\Omega = \{z : \|z\| = 1\} \cup \{re^{i\pi/4} : r \in (1,2)\} \cup \{re^{3i\pi/4} : r \in (1,2)\} \cup \{re^{-3i\pi/4} : r \in (1,2)\} \cup \{re^{-i\pi/4} : r \in (1,2)\}.$$

In accordance with Assumption 5.1, we have

$$\Gamma_1 = \frac{1}{2} + \xi^{-1/2}\Omega \quad \text{and} \quad \Gamma_2 = -\frac{1}{2} + \xi^{-1/2}\Omega,$$

with the jump matrices defined according to Figure 5. Paths of steepest descent are now local paths of steepest descent.

7.2 Uniform Approximation

We have isolated the RHP near the stationary points, and constructed a numerical algorithm to solve the deformed RHP. We show that this numerical algorithm approximates the true solution to the RHP. In order to analyze the error, we introduce the local model problem for this RHP following [7].

Define the Wronskian matrix of parabolic cylinder functions $D_\nu(\zeta)$,

$$Z_0(\zeta) = \begin{bmatrix} D_{-\nu-1}(i\zeta) & D_\nu(\zeta) \\ \frac{d}{d\zeta}D_{-\nu-1}(i\zeta) & \frac{d}{d\zeta}D_\nu(\zeta) \end{bmatrix}, \tag{7.2}$$

and the constant matrices

$$H_{k+2} = e^{i\pi(\nu+1/2)\sigma_3}H_k e^{i\pi(\nu+1/2)\sigma_3}, H_0 = \begin{bmatrix} 1 & 0 \\ h_0 & 1 \end{bmatrix}, H_2 = \begin{bmatrix} 1 & h_1 \\ 0 & 1 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$h_0 = -i\frac{\sqrt{2\pi}}{\Gamma(\nu+1)}, \quad h_1 = \frac{\sqrt{2\pi}}{\Gamma(-\nu)}e^{i\pi\nu}.$$

The sectionally holomorphic function $Z(\zeta)$ is defined as

$$Z(\zeta) = \begin{cases} Z_0(\zeta), & \text{if } \arg\zeta \in (-\pi/4, 0), \\ Z_1(\zeta), & \text{if } \arg\zeta \in (0, \pi/2), \\ Z_2(\zeta), & \text{if } \arg\zeta \in (\pi/2, \pi), \\ Z_3(\zeta), & \text{if } \arg\zeta \in (\pi, 3\pi/2), \\ Z_4(\zeta), & \text{if } \arg\zeta \in (3\pi/2, 7\pi/4). \end{cases}$$

This is used to construct the local solutions

$$\hat\Psi^r(z) = B(z)(-h_1/s_3)^{-\sigma_3/2}e^{it\sigma_3/2}2^{-\sigma_3/2}\begin{bmatrix} \zeta(z) & 1 \\ 1 & 0 \end{bmatrix} Z(\zeta(z))(-h_1/s_3)^{\sigma_3/2}.$$
$$\hat\Psi^l(z) = \sigma_2\hat\Psi^r(-z)\sigma_2,$$

where

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, B(z) = \left(\zeta(z)\frac{z+1/2}{z-1/2}\right)^{\nu\sigma_3}, \zeta(z) = 2\sqrt{-t\theta(z)+t\theta(1/2)}.$$

Consider the sectionally holomorphic matrix-valued function $\hat\Psi(z)$ defined by

$$\hat\Psi(z) = \begin{cases} P(z), & \text{if } |z \pm 1/2| > R, \\ \hat\Psi^r(z), & \text{if } |z - 1/2| < R, \\ \hat\Psi^l(z), & \text{if } |z + 1/2| < R. \end{cases}$$

We use $[\hat G; \hat\Gamma]$ to denote the RHP solved by $\hat\Psi$. See the top panel of Figure 6 for $\hat\Gamma$. In [7] it is shown that Ψ^r satisfies the RHP for Φ exactly near $z = 1/2$ and for Ψ^l near $z = -1/2$. Notice that $\hat\Psi^r$ and $\hat\Psi^l$ are bounded near $z = \pm 1/2$. In the special case where $\log d_1 \in \mathbb{R}$, P remains bounded at $\pm 1/2$. Following the analysis in [7] we write

$$\Phi(z) = \chi(z)\hat\Psi(z),$$

where $\chi \to I$ as $\zeta \to \infty$.

We deform the RHP for $\hat\Psi$ to open up a small circle of radius r near the origin as in Figure 5. We use $[\hat G_1; \hat\Gamma_1]$ to denote this deformed RHP and solution $\hat\Psi_1$. See Figure 6 for $\hat\Gamma_1$. Also it follows that $\hat\Psi(z)P^{-1}(z)$ is uniformly bounded in z and ξ. Further, $\hat\Psi_1$ has the same properties. Since $\hat\Psi_1$ is uniformly bounded in both z and ξ we use (5.6) to show that $\mathcal{C}[\hat G_1; \hat\Gamma_1]^{-1}$ has uniformly bounded norm. We wish to use this to

show the uniform boundedness of the inverse $\mathcal{C}[G;\Gamma]^{-1}$. To do this we extend the jump contours and jump matrices in the following way. Set $\Gamma_e = \Gamma \cup \hat{\Gamma}_1$ and define

$$G_e(z) = \begin{cases} G(z) & \text{if } z \in \Gamma, \\ I & \text{otherwise}, \end{cases}$$

$$\hat{G}_e(z) = \begin{cases} \hat{G}_1(z) & \text{if } z \in \hat{\Gamma}_1, \\ I & \text{otherwise}. \end{cases}$$

The estimates in [7] show that $G_e - \hat{G}_e \to 0$ uniformly on Γ_e. It follows that $\mathcal{C}[\hat{G}_e;\Gamma_e]^{-1}$ is uniformly bounded since the extended operator is the identity operator on $\Gamma\backslash\hat{\Gamma}_1$. Theorem 3.2 implies that $\mathcal{C}[G_e;\Gamma_e]^{-1}$ is uniformly bounded for sufficiently large ξ, which implies that $\mathcal{C}[G;\Gamma]^{-1}$ is uniformly bounded for ξ sufficiently large, noting that the extended operator is the identity operator on the added contours. We now use this construction to prove the uniform convergence of the numerical method using both direct and indirect estimates.

7.3 Application of Direct Estimates

We proceed to show that the RHP for Ψ satisfies the properties of a numerical parametrix. This requires that the jump matrices have uniformly bounded Sobolev norms. The only singularities in the jump matrices is of the form

$$s(z) = \left(\frac{z-1/2}{z+1/2}\right)^{iv}, v \in \mathbb{R}.$$

After transforming to a local coordinate k, $z = \xi^{-1/2}k - 1/2$, we see that

$$S(k) = s(\xi^{-1/2}k - 1/2) = \xi^{-iv/2}\left(\frac{\xi^{-1/2}k+1}{k}\right)^{iv}.$$

The function $S(k)$ is smooth and has uniformly bounded derivatives provided k is bounded away from $k = 0$. The deformations applied thus far guarantee that k will be bounded away from 0. To control behavior of the solution for large k we look at the exponent which appears in the jump matrix

$$\theta(z) = \frac{2i}{3} - 4i\left(z+\frac{1}{2}\right)^2 + \frac{8i}{3}\left(z+\frac{1}{2}\right)^3,$$

and define

$$\Theta(k) = \theta(\xi^{-1/2}k - 1/2) = \frac{2i}{3} - 4ik^2/\xi + \frac{8i}{3}k^3/\xi^{3/2}.$$

If we assume that the contours are deformed along the local paths of steepest descent, all derivatives of $e^{\xi\Theta(k)}$ are exponentially decaying, uniformly in ξ. After applying the same procedure at $z = 1/2$ and after contour truncation, Theorem 5.1 implies the RHP for Ψ satisfies the hypotheses of Theorem 5.1, proving strong uniform convergence.

7.4 Application of Indirect Estimates

The second approach is to use the solution of the model problem to construct an numerical parametrix. Since we have already established strong uniform convergence we proceed to establish a theoretical link with the method of nonlinear steepest descent, demonstrating that the success of nonlinear steepest descent implies the success of the numerical method, even though the numerical method does not depend on the details of the nonlinear steepest descent. We start with the RHP $[\hat{G}_1;\hat{\Gamma}_1]$ and its solution $\hat{\Psi}_1$. As before, see Figure 6 for $\hat{\Gamma}_1$. Define $\hat{u} = (\hat{\Psi}_1)^+ - (\hat{\Psi}_1)^-$ which is the solution of the associated SIE on $\hat{\Gamma}_1$. The issue here is that we cannot scale the deformed RHP in Figure 5 so that it is posed on the same contour as $[G;\Gamma]$. We need to remove the larger circle.

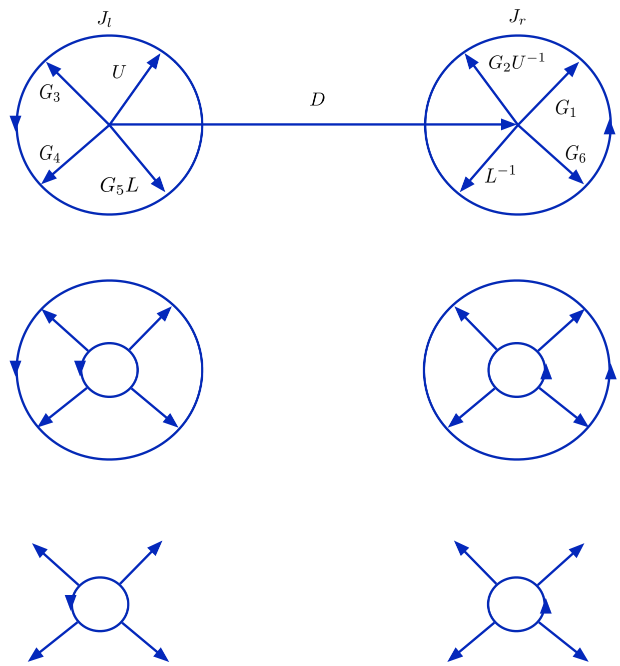

Figure 6: Top: Jump contours for the model problem with solution $\hat{\Psi}$. Note that J_r and J_l are the jumps on the outside of the circles. They tend uniformly to the identity as $\xi \to \infty$ [7]. Center: The jump contours, $\hat{\Gamma}_1$, for the function $\hat{\Psi}_1$. The inner circle has radius r and the outer circle has radius R. Bottom: Contour on which $\hat{U}$ is non-zero. This can be matched up with the right contour in Figure 5.

We define a new function $\hat{U} = \hat{u}\phi$ where ϕ is a C^∞ function with support in $(B(-1/2, R) \cup B(1/2, R)) \cap \hat{\Gamma}_1$ such that $\phi = 1$ on $(B(1/2, r) \cup B(-1/2, r)) \cap \hat{\Gamma}_1$ for $r < R$. Let $\hat{\Gamma}_2$ be the support of $\hat{U}$ (see bottom contour in Figure 6). Define $\hat{\Psi}_2 = I + \mathcal{C}_{\hat{\Gamma}_2} \hat{U}$. From the estimates in [7], it follows that

$$\hat{\Psi}_2^+ = \hat{\Psi}_2^- \hat{G}_2$$

where $\hat{G}_2 - G$ tends uniformly to zero as $\xi \to \infty$. We have to establish the required smoothness of $\hat{U}$. We do this explicitly from the above expression for $\hat{\Psi} P^{-1}$ after using the scalings $z = \xi^{-1/2} k \pm 1/2$. The final step is to let ξ be large enough so that we can truncate both $[G; \Gamma]$ and $[\hat{G}_2; \hat{\Gamma}_2]$ to the same contour. We use Proposition 5.2 to prove that this produces a numerical parametrix. Additionally, this shows how the local solution of RHPs can be tied to stable numerical computations of solutions.

Remark 7.1. *This analysis relies heavily on the boundedness of P. These arguments would fail if we were to let P have unbounded singularities. In this case one approach would be to solve the RHP for χ. The jump for this RHP tends to the identity. To prove weak uniformity for this problem one only needs to consider the trivial RHP with the jump being the identity matrix as a numerical parametrix.*

7.5 Numerical Results

In Figure 7 we plot the solution to Painlevé II with $(s_1, s_2, s_3) = (1, -2, 3)$ and demonstrate numerically that the computation remains accurate in the asymptotic regime. We use $u(n, x)$ to denote the approximate solution obtained with n collocation points per contour. Since we are using (7.1) we consider the estimated relative error by dividing the absolute error by $\sqrt{x}$. We see that we retain relative error as x becomes large.

Remark 7.2. *Solutions to Painlevé II often have poles on the real line, which correspond to the RHPs not having a solution. In other words, $\|\mathcal{C}[\Gamma, \Omega]^{-1}\|$ is not uniformly bounded, which means that the theory of this paper does not apply. However, the theorems can be adapted to the situation where x is restricted to a*

subdomain of the real line such that $\|\mathcal{C}[\Gamma,\Omega]^{-1}\|$ is uniformly bounded. This demonstrates asymptotic stability of the numerical method for solutions with poles, provided that x is bounded away from the poles, similar to the restriction of the asymptotic formulæ in [7].

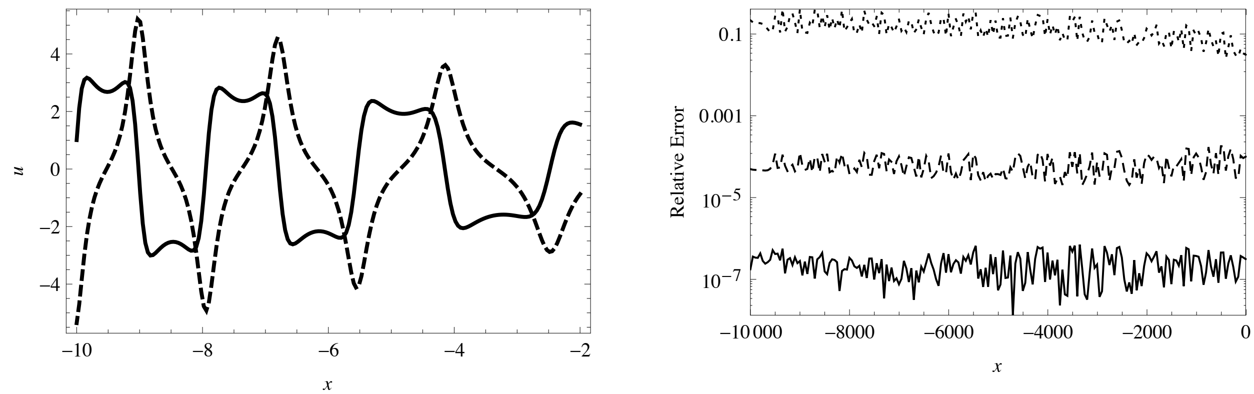

Figure 7: Left: Plot of the solution, u, for small x (Solid: real part, Dashed: imaginary part). Right: Relative error. Solid: $|x|^{-1/2}|u(12,x)-u(36,x)|$, Dashed: $|u(8,x)-u(36,x)|/\sqrt{|x|}$, Dotted: $|u(4,x)-u(36,x)|/\sqrt{|x|}$. This plot demonstrates both uniform approximation and spectral convergence.

8 Application to the modified Korteweg–de Vries Equation

In this section we consider the numerical solution of the modified Korteweg-de Vries equation (1.2) (mKdV) for $x<0$. The RHP for mKdV is [15]

$$\begin{aligned}
\Phi^+(z) &= \Phi^-(z)G(z), \quad z\in\mathbb{R},\\
\Phi(\infty) &= I,\\
G(z) &= \begin{bmatrix} 1-\rho(z)\rho(-z) & -\rho(-z)e^{-\theta(z)} \\ \rho(z)e^{\theta(z)} & 1\end{bmatrix},\\
\theta(z) &= 2izx+8iz^3t.
\end{aligned}$$

In the cases we consider ρ is analytic in a strip that contains $\mathbb{R}$. If $x\ll -ct^{1/3}$ the deformation is similar to the case considered above for Painlevé II and asymptotic stability follows by the same arguments. We assume $x=-12c^2t^{1/3}$ for some positive constant c. This deformation is found in [15]. We rewrite θ:

$$\theta(z) = -24ic^2(zt^{1/3}) + 8i(zt^{1/3})^3.$$

We note that $\theta'(z_0)=0$ for $z_0=\pm\sqrt{-x/(12t)}=\pm ct^{-1/3}$. We introduce a new variable $k=zt^{1/3}/c$ so that

$$\theta(kct^{-1/3}) = -24ic^3k + 8ic^3k^3 = 8ic^3(k^3-3k).$$

For a function of $f(z)$ we use the scaling $\tilde{f}(k)=f(kct^{-1/3})$. The functions $\tilde{\theta}$, $\tilde{G}$ and $\tilde{\rho}$ are identified similarly. After deformation and scaling, we obtain the following RHP for $\tilde{\Phi}(k)$:

$$\begin{aligned}
\tilde{\Phi}^+(k) &= \tilde{\Phi}^-(k)J(k), k\in\Sigma=[-1,1]\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4,\\
J(k) &= \begin{cases} \tilde{G}(k), & \text{if } k\in[-1,1],\\ \begin{bmatrix} 1 & 0 \\ \tilde{\rho}(k)e^{\tilde{\theta}(k)} & 1\end{bmatrix}, & \text{if } k\in\Gamma_1\cup\Gamma_2,\\ \begin{bmatrix} 1 & -\tilde{\rho}(-k)e^{-\tilde{\theta}(k)} \\ 0 & 1\end{bmatrix}, & \text{if } k\in\Gamma_3\cup\Gamma_4,\end{cases}
\end{aligned}$$

where Γ_i, $i = 1, 2, 3, 4$, shown in Figure 8, are locally deformed along the path of steepest descent. To reconstruct the solution to mKdV we use the formula

$$u(x,t) = 2iz_0 \lim_{k\to\infty} k\tilde{\Phi}_{12}(k). \tag{8.1}$$

Remark 8.1. *We assume ρ decays rapidly at ∞ and is analtytic in a strip that contains the real line. This allows us to perform the initial deformation which requires modification of the contours at ∞. As t increases, the analyticity requirements on ρ are reduced; the width of the strip can be taken to be smaller if needed. We only require that each Γ_i lies in the domain of analticity for $\tilde{\rho}$. More specifically, we assume t is large enough so that when we truncate the contours for numerical purposes, they lie within the strip of analyticity for $\tilde{\rho}$.*

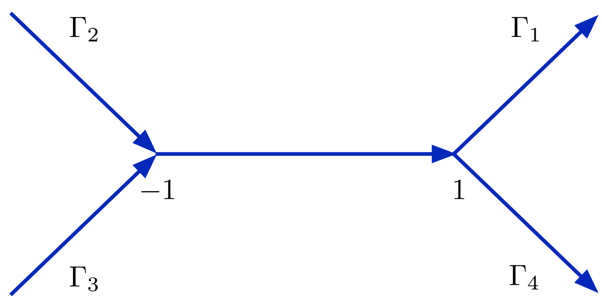

Figure 8: Jump contours for the RHP for mKdV.

The parametrix derived in [5] is used to show that $\mathcal{C}[J,\Sigma]$ has an inverse that is uniformly bounded by using (5.6) as was done in the previous section. We use the analyticity and decay of ρ at ∞ along with the fact that the contours pass along the paths of steepest descent.

The contour is fixed (i.e., independent of x, t and c), and this situation is more straightforward to analyze than the previous example. Repeated differentiation of $J(k)$ proves that this deformation yields a uniform numerical approximation. Furthermore, replacing c by any smaller value yields the same conclusion. This proves the uniform approximation of mKdV in the Painlevé region

$$\{(x,t) : t \geq \epsilon, \quad x \leq -\epsilon, x \geq -12c^2t^{1/3}\}, \quad \epsilon > 0.$$

where ϵ is determined by the analyticity of ρ.

8.1 Numerical Results

In Figure 9 we show the solution with initial data $u(x,0) = -2e^{-x^2}$ with $c = \sqrt{9/4}$. The reflection coefficient is obtained using the method described in [15]. We use the notation $u(n,x,t)$ to denote the approximate solution obtained with n collocation points per contour. We see that the absolute error tends to zero rapidly. More importantly, the relative error remains small. We approximate the solution uniformly on the fixed, scaled contour. When we compute the solution using (8.1) we multiply by z_0 which is decaying to zero along this trajectory. This is how the method maintains accuracy even when comparing relative error.

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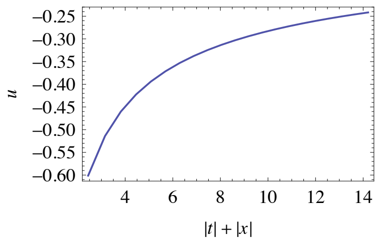

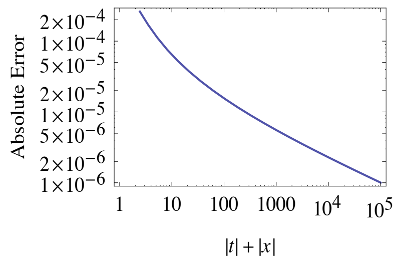

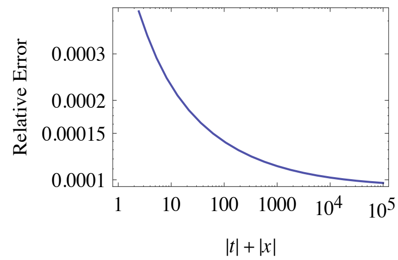

Figure 9: Left: Plot of the solution along $x = -(3t)^{1/3}$ for small time. Center: Absolute error, $|u(5,x,t) - u(10,x,t)|$, for long time Right: Relative error $|u(5,x,t) - u(10,x,t)|/|u(10,x,t)$ for long time.

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